Modelling Card Shuffling as a Markov Chain

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March 31, 2026

Part I

Introduction

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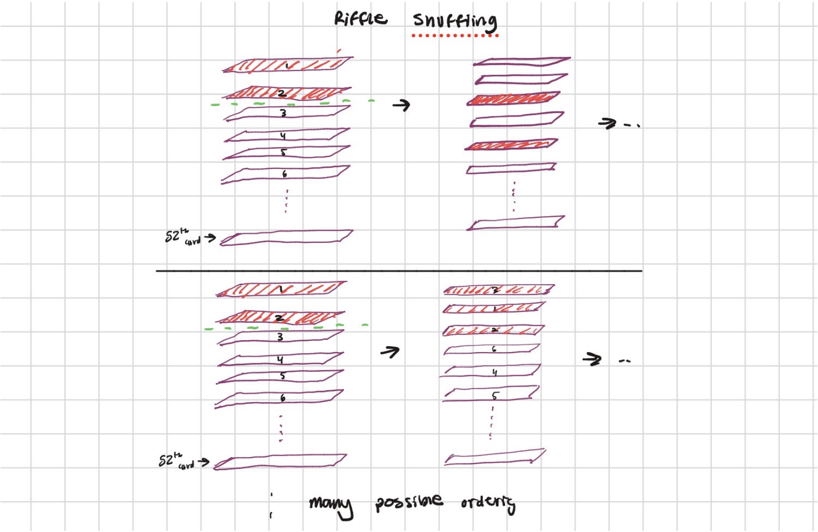
To do:

1. Determine the distribution of orderings of cards after many shuffles
2. Show that the distribution approaches Uniform Distribution

Riffle Shuffling



Definition of a Markov Chain

Definition

A sequence of random variables $\{X_n\}_{n\geq 0}$ with state space S is a **Markov chain** with transition matrix $P \in [0,1]^{S\times S}$ if, for all $n \geq 0$ and all states $x_0, \ldots, x_{n-1}, x, y \in S$,

$$\mathbb{P}(X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = P_{xy}$$

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Interpretation:

- The next step depends only on the **current state**
- It does **not** depend on the full history
- This property is commonly called the **Markov Property**

Example

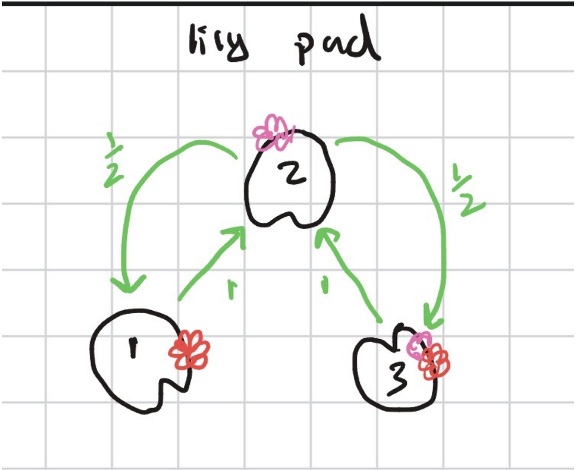
An example of a Markov chain is:

1. "Lily Pad" Frog jump

   State Space: $\{1, 2, 3\}$

   The frog must jump from 1 to 2 and from 3 to 2 with probability 1, but can jump to either 1 or 3 with probability $\frac{1}{2}$, if it's on the second lily pad.

"Lily pad" Frog jump



Riffle Shuffling is a Markov Chain

Shuffling can be considered as a Markov chain on orderings of a deck of cards.

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Let S_{52} be the set of all permutations of a 52-card deck, and let X_n be the ordering after n shuffles. Then $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix

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Reframing our previous questions:

1. What is the distribution of X_n after many shuffles?
2. Does X_n converge to a uniform distribution over S?

Part II

Terminology and the Convergence Theorem

Stationary Distribution

Definition

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This means:

- if the chain starts with distribution π,
- then after one step it still has distribution π,
- and after any number of steps it remains π.

Stationary Distribution for Shuffling

Let S_{52} be the set of all possible orderings of a 52-card deck.

Consider the **uniform distribution**

$$\pi(x) = \frac{1}{52!} \qquad \text{for every ordering } x \in S.$$

For card shuffling:

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Key idea:

- The uniform distribution is the stationary distribution for common shuffle models
- So a "well-shuffled" deck should look uniform

Does repeated shuffling make the distribution approach uniform?

Irreducibility

Definition

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space S and transition matrix P. We say the chain is **irreducible** if for all $x, y \in S$, there exists $n > 0$ such that

$$P^n_{xy} > 0$$

That is, every state y can be reached from every state x in some number of steps. If there exist $x, y \in S$ such that $P^n_{xy} = 0$ for all $n > 0$, the chain is **reducible**.

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For riffle shuffling, at each step the deck is split into 2 parts and interleaved, where each splitting and interleaving occurs with positive probability. This allows us to construct a sequence of shufflings to transition between any two orderings.

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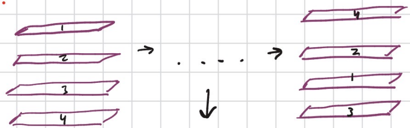
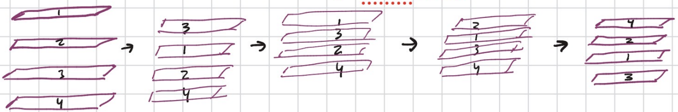
Therefore, $P_{xy}^{n} > 0$ for some $n > 0$, for all $x, y \in S$.

Irreducibility with riffle shuffling

example



theres alwas a positive posibility of taking one card and adding it on top while the rest of the cards stay stationary



Aperiodicity

Definition

Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space S. Let $x \in S$ with the property that $P^n_{xx} > 0$ for some $n \geq 0$. Define:

$$c(x) := gcd\{n \geq 1 : P^n_{xx} > 0\}$$

where gcd stands for greatest common divisor.

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a. If $c(x) \geq 2$, the state $x \in S$ is called **periodic** with **period** $c(x)$
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a. If $c(x) \geq 2$, the state $x \in S$ is called **periodic** with **period** $c(x)$
b. If $c(x) = 1$, then state x is called **aperiodic**
c. *If all states in S have the same period c, we call the Markov chain periodic with period c*
d. *If all states in S are aperiodic, we call the Markov Chain aperiodic*

Examples

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In the riffle shuffle, you have to cut the deck randomly in two decks, but with positive probability, you can place them back together in the same order. Therefore, $P_{xx} > 0$ implying the gcd would be 1 which is the condition to satisfy Aperiodicity.

Convergence Theorem

Theorem

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space S, transition matrix P_{xy}, and stationary distribution π. If the chain is **irreducible** *and* **aperiodic**, *then for all $x, y \in S$,*

$$P_{xy}^n \to \pi(y) \quad \text{as } n \to \infty$$

In particular, π is the **unique** *stationary distribution.*

For Riffle shuffling: Since it is irreducible, aperiodic and the stationary distribution is the uniform distribution, X_n converges to the uniform distribution over S_{52}.

Part III

How fast is this Convergence?

Some additional terminology

The number of shuffles required to approach Uniform Distribution is called the Mixing Time.

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We measure the total distance away from the Uniform Distribution as the Total Variation Distance (TV)

Definition

Let π and η be distributions on the state space S. Then, TV is defined as:

$$\|\pi - \eta\|_{TV} = \frac{1}{2}\sum_{x \in S} |\pi(x) - \eta(x)|$$

Theorem for Mixing Time

Theorem (Aldous and Diaconis, 1986 [1])

Let $\{X_t\}_{t \geq 0}$ be the riffle shuffle Markov chain on n cards. Then for all $k \geq 0$,

$$||X_k - U||_{TV} \leq 1 - \prod_{j=1}^{n-1}(1 - \frac{j}{2^k})$$

If $k = 2\log_2 \frac{n}{c}$ for an arbitrary $c > 0$ then:

$$1 - \prod_{j=1}^{n-1}(1 - \frac{j}{2^k}) \to \frac{1}{2}c^2, \quad \text{as } n \to \infty,\ c \to 0$$

Remark:

This mixing time can be further improved to $\frac{3}{2}\log_2 n$ [2].

Part IV

Simulation

Simulation Setup and Measure

We simulate repeated **riffle shuffles** on a small deck.

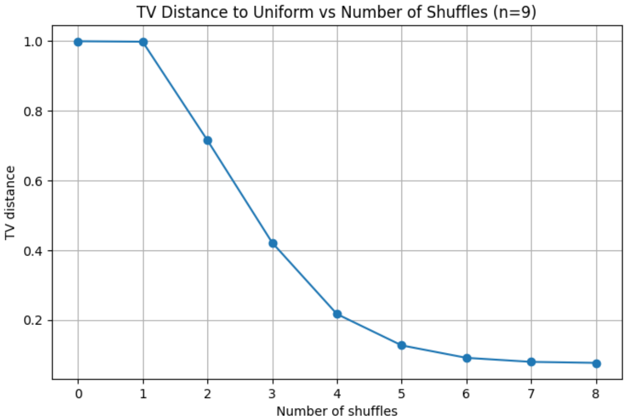
Setup:

- $n = 9$ cards $\Rightarrow |S_9| = 362,880$ orderings
- Start from $(1,2,3,4,5,6,7,8,9)$
- Let π_k be the distribution after k shuffles
- Estimate π_k using many trials

Goal: compare π_k to the uniform distribution U

- $\|\pi_k - U\|_{TV} \approx 1$ not well shuffled
- $\|\pi_k - U\|_{TV} \approx 0$ close to uniform

Simulation Result



- n = 9 (# of cards)
- trials = 5,000,000

Observation:

- As the number of shuffles k increases, the TV distance decreases
- The distribution of the deck becomes closer to uniform

Conclusion:

Repeated shuffling $\longrightarrow$ distribution becomes close to uniform

Thank you!

Part V

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