Creative Telescoping and Automatic Proofs of Hypergeometric Identities

CO-3 Final Presentation

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I. Introduction

Why Do We Need Algorithms for Sums?

- How do we evaluate complicated sums?
- Closed-form vs numerical computation
- Can this be done algorithmically?

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

Goal: turn summation into a systematic process

What are Canonical and Normal Forms?

- Canonical forms
  - Standardized representation where all objects in the class are described in one way (one-to-one)
  - Common example: $A = PDP^{-1}$
- Normal Forms
  - Standardized representation where the form is simplified or standardized however may not be unique. (additive property is preserved)
  - Common Example: polynomials

How Can These Forms Generate Proof Machines?

- Polynomial Identities
  - Using Maple function `expand` to convert to normal form to prove polynomial identities
- Trig Identities
  - Let $w := \exp(ix)$, thus the equality of rational trig forms can be reduced in w using $sin(x) = \frac{w - w^{-1}}{2i}$ and $cos(x) = \frac{w + w^{-1}}{2}$
  - Express everything in terms of $arcsin$ by setting $z = sin(w)$ and $cos(x) = \sqrt{1 - sin^2(w)}$. Differentiating with respect to one of the variables and using $arcsin$ identities to prove.
- …And many more…

II. Hypergeometric Functions and Identities

What are Hypergeometric Series?

- Series $\sum_{k\geq 0}^{n} t_k$ which $t_0 = 1$ where the ratio of the two consecutive terms is a rational function of the summation index k in which

$$\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)}$$

- Many functions are hypergeometric this includes exponential, logarithmic, trigonometric and binomial.
- Is there a standard way to describe <u>all</u> hypergeometric functions?

How to Convert a Function into ${}_pF_q$ notation

Let polynomials P and Q be completely factored thus

$$\frac{t_{k+1}}{t_k} \overset{def}{=} \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x$$

where x is a constant. By normalizing with $t_0 = 1$ then the hypergeometric series of the terms t_k's such as the series $\sum_{k\geq 0} t_k x^k$ is written as

$${}_pF_q\left[\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}; x\right]$$

which the a's are the upper parameters and the b's are the lower parameters (non-negative/0)

The initial term is 1, and the ratio of the $(k+1)^{\text{st}}$ term to the k^{th} is given above for all $k \geq 0$

How to Determine if a Given Function is Hypergeometric?

Use the Hypergeometric Series Lookup Algorithm:

1. Given a series $\sum_k t_k$: Shift summation index k so that the sum starts at $k=0$ with a nonzero term. Factor the term corresponding to $k=0$ as a common factor so that the first term of the sum will be 1
2. Simplify the ratio $\frac{t_{k+1}}{t_k}$ to $\frac{P(k)}{Q(k)}$ where P and Q are polynomials.
3. Completely factor P and Q into linear factors and write the ratio in the form

$$\frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x$$

4. Using the common factor in step 1, multiply that with the hypergeometric series

$$ {}_pF_q\left[\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}; x\right]$$

What is the Hypergeometric Database and Does It Exist?

- Examples from the “database”:
    - **Gauss’s ${}_2F_1$ identity:** where b is non-positive or $c-a-b$ has a positive real part then

$$ {}_2F_1\left[\begin{matrix} a & b \\ c & \end{matrix} ; 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} $$

    - **Dixon’s ${}_2F_1$ identity:** if $a-b+c=1$

$$ {}_2F_1\left[\begin{matrix} a & b \\ c & \end{matrix} ; -1\right] = \frac{\Gamma(\frac{b}{2}+1)\Gamma(b-a+1)}{\Gamma(b+1)\Gamma(\frac{b}{2}-a+1)} $$

if b is a negative integer

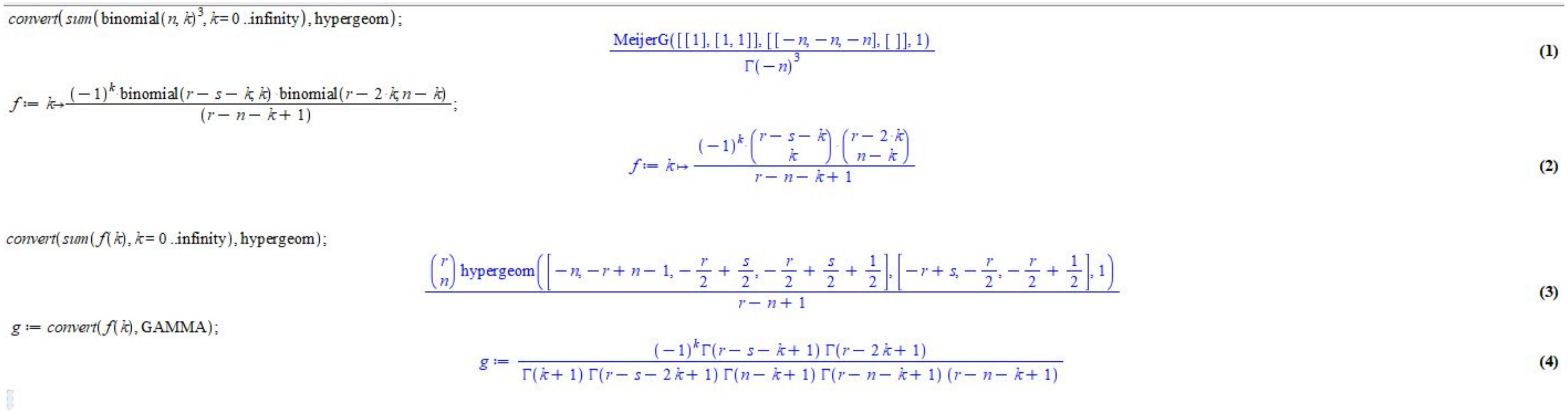
$$ {}_2F_1\left[\begin{matrix} a & b \\ c & \end{matrix} ; -1\right] = 2\cos\left(\frac{\pi b}{2}\right)\frac{\Gamma(|b|)\Gamma(b-a+1)}{\Gamma(\frac{|b|}{2})\Gamma(\frac{b}{2}-a+1)} $$

    - **Saalschutz’s ${}_3F_2$ identity:** if $d+e=a+b+c+1$ and c is a negative integer

$$ {}_3F_2\left[\begin{matrix} a & b & c \\ d & e & \end{matrix} ; 1\right] = \frac{(d-a)_{|c|}(d-b)_{|c|}}{d_{|c|}(d-a-b)_{|c|}} $$

- However the Hypergeometric DOES NOT and CANNOT exist because
    - No known algorithm to transform a given sum into a specific identity, a sum could be a sequence of infinite transformations and extreme complexity.

An Example of How to “Evaluate” a Hypergeometric Sum



$\mathit{convert}(\mathit{sum}(\mathrm{binomial}(n, k)^3, k = 0\,..\mathrm{infinity}), \mathrm{hypergeom});$

$$\frac{\mathrm{MeijerG}([[1], [1, 1]], [[-n, -n, -n], [\,]], 1)}{\Gamma(-n)^3} \quad (1)$$

$f := k \rightarrow \dfrac{(-1)^k \cdot \mathrm{binomial}(r - s - k, k) \cdot \mathrm{binomial}(r - 2 \cdot k, n - k)}{(r - n - k + 1)};$

$$f := k \mapsto \frac{(-1)^k \cdot \binom{r - s - k}{k} \cdot \binom{r - 2 \cdot k}{n - k}}{r - n - k + 1} \quad (2)$$

$\mathit{convert}(\mathit{sum}(f(k), k = 0\,..\mathrm{infinity}), \mathrm{hypergeom});$

$$\frac{\binom{r}{n} \mathrm{hypergeom}\left(\left[-n, -r + n - 1, -\frac{r}{2} + \frac{s}{2}, -\frac{r}{2} + \frac{s}{2} + \frac{1}{2}\right], \left[-r + s, -\frac{r}{2}, -\frac{r}{2} + \frac{1}{2}\right], 1\right)}{r - n + 1} \quad (3)$$

$g := \mathit{convert}(f(k), \mathrm{GAMMA});$

$$g := \frac{(-1)^k \Gamma(r - s - k + 1)\, \Gamma(r - 2\,k + 1)}{\Gamma(k + 1)\, \Gamma(r - s - 2\,k + 1)\, \Gamma(n - k + 1)\, \Gamma(r - n - k + 1)\, (r - n - k + 1)} \quad (4)$$

- Using `convert/hypergeom` and defining the summand first can make it less tricky
- Note that it is not in ${}_pF_q$ visual notation but in `hypergeom`

III. Finding recurrences for Hypergeometric Sums

Sister Celine’s Algorithm

Goal: Give a sum $f(n) = \sum_k F(n,k)$, find a linear recurrence in n that F(n) satisfies.

What does it do?

- It looks for a recurrence of the form: $\sum_{i=0}^{I} \sum_{j=0}^{J} a_{i,j}(n) f(n-j, k-i) = 0$
- By eliminating k, we get a recurrence purely in n for F(n).

Key Insight: Treat f(n,k) as hypergeometric in both variables, so the ratio f(n,k)/f(n−1,k) and f(n,k)/f(n,k−1) are rational functions.

Sister Celine’s Algorithm - Steps & Theorem

Algorithm Steps:

1. Assume the form: Choose integers I and J (typically start with I=J=1).
2. Divide by f(n,k): This yields a rational function equation in n and k.
3. Clear denominators: Multiply through to obtain a polynomial equation in k and n.
4. Collect coefficients: Treat k as an indeterminate. Equate coefficients of each power k^m to zero.
5. Solve the system: This gives a linear system for the unknown coefficients of $a_{i,j}(n)$.
6. Iterate: If no non-trivial solution exists, increase I and/or J and repeat.

Fundamental Theorem

- If f(n,k) is a proper hypergeometric term, then there exist integers I,J (with I≥J) and polynomials $a_{i,j}(n)$, not all zero, satisfying the mixed recurrence.
- Consequence: The algorithm always terminates. A recurrence for F(n) exists and can be found in finitely many steps.

Sister Celine’s Algorithm: A Maple Demonstration

GOAL: Find a recurrence for $f(n) = \sum_{k=0}^{n} \binom{n}{k}^2$. First, find recurrence for $F(n,k) = \binom{n}{k}^2$.

```
> read("EKHAD.mpl")
```

```
Last update: April 26, 2018, thanks to Daniel G. DuParc
Previous updates:   Dec. 11, 2015 (adding procedures AZdI, AZcI; May 30, 2015 (adding procedure TerryTao )
May 29, 2014 (adding Ekhad )
Version of July 2003: adapted to Maple 8 and 9
Many thanks to Drew Sills
In the penultimate Version of Feb 25, 1999 a suggestion
of Frederic Chyzak was used, with considerable
speed-up. We thank him SO MUCH!

The penpenultimate version, Feb. 1997,
corrected a subtle bug discovered by Helmut Prodinger

Previous versions benefited from comments by Paula Cohen,
Lyle Ramshaw, and Bob Sulanke.

This is EKHAD, One of the Maple packages
accompanying the book
"A=B"
(published by A.K. Peters, Wellesley, 1996)
by Marko Petkovsek, Herb Wilf, and Doron Zeilberger.

The most current version is available on WWW at:
http://www.math.rutgers.edu/~zeilberg .
Information about the book, and how to order it, can be found in
http://www.central.cis.upenn.edu/~wilf/AeqB.html .
Please report all bugs to: zeilberg@math.rutgers.edu .
All bugs or other comments used will be acknowledged in future
versions.

For general help, and a list of the available functions,
type "ezra();". For specific help type "ezra(procedure_name)"
```

$$TerryTao\left(x^2+1, \frac{1}{\sqrt{x^2+1}}, x, z, k, 2\,k, K\right) \qquad (4)$$

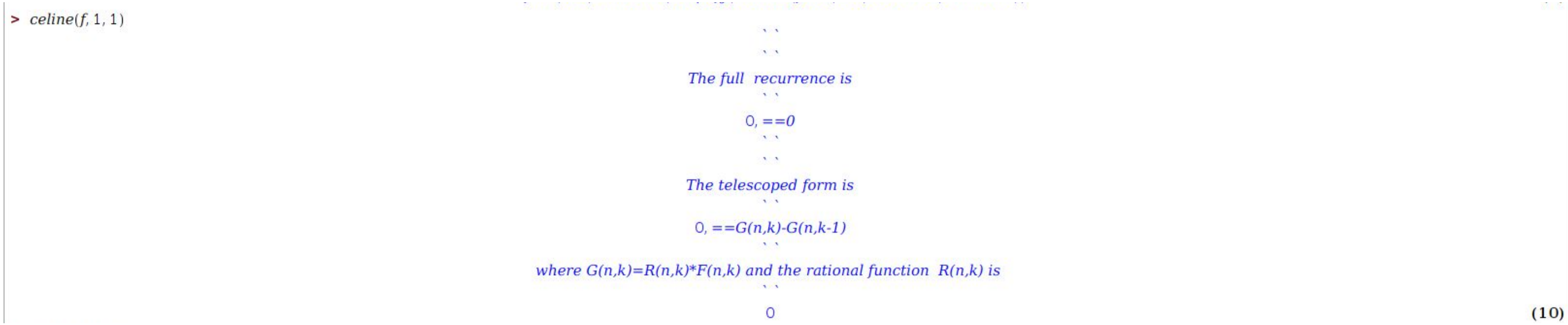
Sister Celine’s Algorithm: A Maple Demonstration

First, convert our input into something that celine() can process:

```
> fraw := (n, k) → binomial(n, k)^2 :

  f := (n, k) → normal( simplify( convert(fraw(n, k), factorial), factorial ) );
                     f := (n, k) ↦ normal(simplify(convert(fraw(n, k), factorial), factorial))        (9)
```

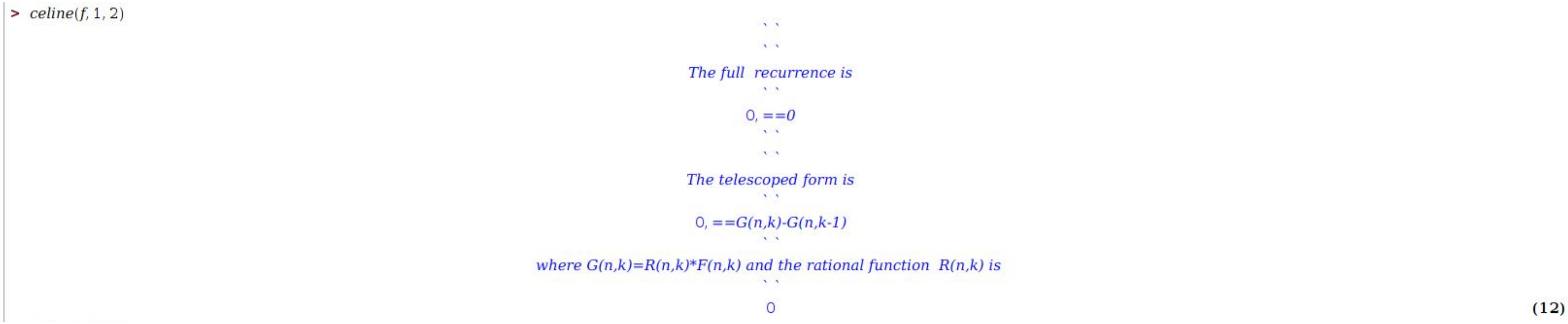
Now, try calling celine() with (n,k) = (1,1):



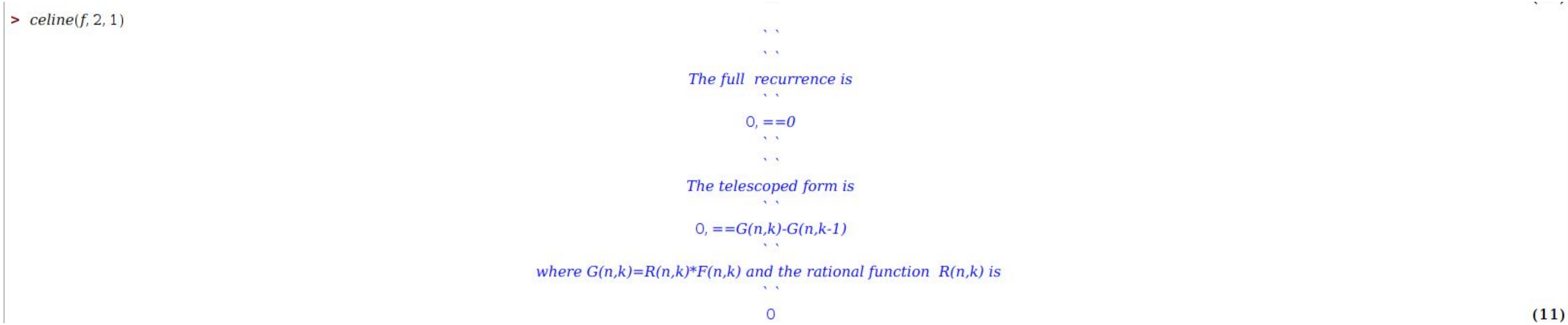
```
> celine(f, 1, 1)
                                        ` `
                                        ` `
                               The full  recurrence is
                                        ` `
                                      0, ==0
                                        ` `
                                        ` `
                               The telescoped form is
                                        ` `
                                 0, ==G(n,k)-G(n,k-1)
                                        ` `
               where G(n,k)=R(n,k)*F(n,k) and the rational function  R(n,k) is
                                        ` `
                                         0                                                   (10)
```

We get the trivial recurrence… no good.

Sister Celine's Algorithm: A Maple Demonstration

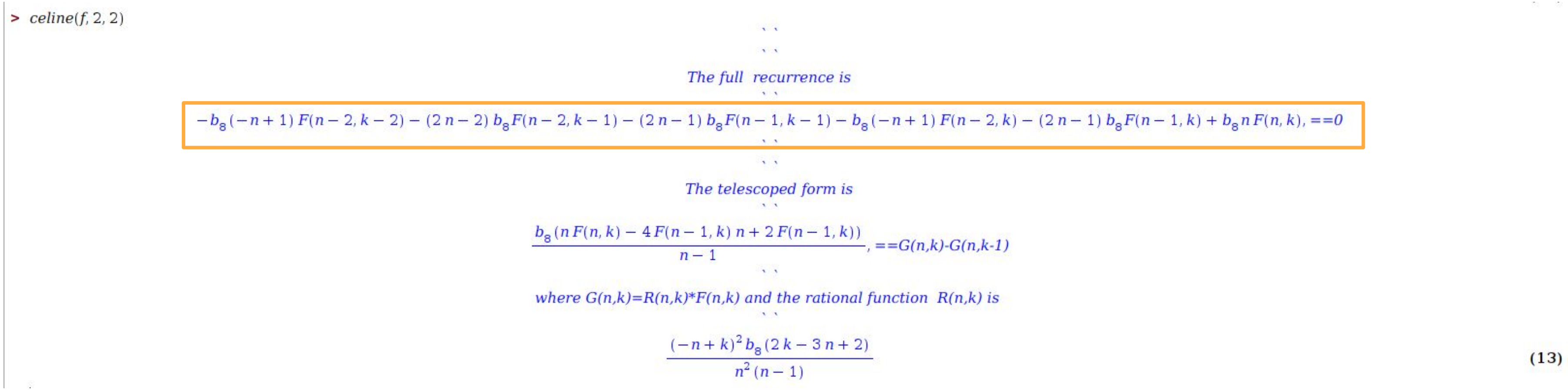


```
> celine(f, 1, 2)
                                ` `
                                ` `
                      The full  recurrence is
                                ` `
                               0, ==0
                                ` `
                                ` `
                      The telescoped form is
                                ` `
                        0, ==G(n,k)-G(n,k-1)
                                ` `
     where G(n,k)=R(n,k)*F(n,k) and the rational function  R(n,k) is
                                ` `
                                 0                                  (12)
```



```
> celine(f, 2, 1)
                                ` `
                                ` `
                      The full  recurrence is
                                ` `
                               0, ==0
                                ` `
                                ` `
                      The telescoped form is
                                ` `
                        0, ==G(n,k)-G(n,k-1)
                                ` `
     where G(n,k)=R(n,k)*F(n,k) and the rational function  R(n,k) is
                                ` `
                                 0                                  (11)
```

Sister Celine’s Algorithm: A Maple Demonstration



> celine(f, 2, 2)

The full recurrence is

$$-b_8(-n+1)F(n-2,k-2)-(2n-2)b_8F(n-2,k-1)-(2n-1)b_8F(n-1,k-1)-b_8(-n+1)F(n-2,k)-(2n-1)b_8F(n-1,k)+b_8nF(n,k),\ ==0$$

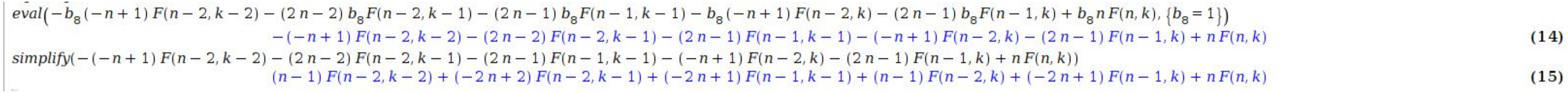
The telescoped form is

$$\frac{b_8(nF(n,k)-4F(n-1,k)n+2F(n-1,k))}{n-1},\ ==G(n,k)\text{-}G(n,k\text{-}1)$$

*where G(n,k)=R(n,k)*F(n,k) and the rational function R(n,k) is*

$$\frac{(-n+k)^2 b_8(2k-3n+2)}{n^2(n-1)} \tag{13}$$

Take b8=1, collect similar terms and simplify:



$eval(-b_8(-n+1)F(n-2,k-2)-(2n-2)b_8F(n-2,k-1)-(2n-1)b_8F(n-1,k-1)-b_8(-n+1)F(n-2,k)-(2n-1)b_8F(n-1,k)+b_8nF(n,k),\{b_8=1\})$

$$-(-n+1)F(n-2,k-2)-(2n-2)F(n-2,k-1)-(2n-1)F(n-1,k-1)-(-n+1)F(n-2,k)-(2n-1)F(n-1,k)+nF(n,k) \tag{14}$$

$simplify(-(-n+1)F(n-2,k-2)-(2n-2)F(n-2,k-1)-(2n-1)F(n-1,k-1)-(-n+1)F(n-2,k)-(2n-1)F(n-1,k)+nF(n,k))$

$$(n-1)F(n-2,k-2)+(-2n+2)F(n-2,k-1)+(-2n+1)F(n-1,k-1)+(n-1)F(n-2,k)+(-2n+1)F(n-1,k)+nF(n,k) \tag{15}$$

Sum over k to get a recurrence for f(n)… and things cancel:

$$\cancel{(n-1)f(n-2)}+\cancel{(-2n+2)f(n-2)}+(-2n+1)f(n-1)+\cancel{(n-1)f(n-2)}+(-2n+1)f(n-1)+nf(n)=0.$$

Sister Celine’s Algorithm: A Maple Demonstration

Simplifying,

$$\boxed{2(1-2n)f(n-1)+nf(n)=0.}$$

Zeilberger’s Algorithm

The Problem: Sister Celine’s algorithm works, but it is often slow and requires solving large linear systems.

The Solution:

- A fast alternative
- Outputs a recurrence of the form:

$$\sum_{j=0}^{J} a_j(n)F(n+j) = G(n)$$

- Usually, G(n) telescopes to 0
- **Core Idea:** Instead of eliminating k entirely from f, it uses Gosper’s algorithm to find a certificate R(n,k) such that:

$$\sum_{j=0}^{J} a_j(n)\, f(n+j,k) = g(n,k+1) - g(n,k)$$

- Summing over k makes the right side telescope to 0.

Gosper's Algorithm

- For special case when summand can be written $z_{n+1} - z_n = t_n.$
- Discrete analogue to the "indefinite integration problem."
- Doesn't always work…
- …But lets us combine Steps 1 and 2 when it does.

$$S(x) = \int_a^x f(t)dt \Longleftrightarrow S(n) = \sum_{k=a}^{n} t_k$$

$$F'(t) = f(t) \Longleftrightarrow z_{n+1} - z_n = t_n$$

$$S(x) = F(x) - F(a) \Longleftrightarrow S(n) = z_n - z_a$$

IV. Solving Recurrences

Simple technique: “Unrolling” the recurrence

- Works for simple or first order recurrences

Example:
$$f_n = \frac{2(n+1)}{n} f_{n-1} \ , \quad f(1) = 1$$

Solution:
$$\begin{aligned} f_n &= \frac{2(n+1)}{n} f_{n-1} \\ &= \frac{2(n+1)}{n} \frac{2n}{n-1} f_{n-2} = 2^2 \frac{n+1}{n-1} f_{n-2} \\ &= \frac{2^2(n+1)2(n-1)}{(n-1)(n-2)} f_{n-3} = 2^3 \frac{(n+1)}{(n-2)} f_{n-3} \\ &\vdots \\ &= 2^k \frac{(n+1)}{(n-k+1)} f_{n-k}. \end{aligned}$$

Simple Technique: “Unrolling” the Recurrence

$$\forall k < n, \quad f_n = 2^k \frac{(n+1)}{(n-k+1)} f_{n-k}.$$

Now, make a substitution so that $n - k = 1.$

$n - k = 1 \implies k = n - 1,$ so subbing this into the above expression yields

$f_n = \frac{2^{n-1}(n+1)}{n-(n-1)+1} f_{n-(n-1)} = 2^{n-2}(n+1) f_1$ so $\forall n,$ $\boxed{f_n = 2^{n-2}(n+1).}$

Another example… the sum from before:

$$f(n) = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

$$f(0) = 1$$

Recall our Celine output:

$$\boxed{2(1-2n)f(n-1) + nf(n) = 0.}$$

We can unroll this recurrence to get:

$$f(n) = \frac{2^n(2n-1)(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1}{n!} f(0)$$

$$= 2^n \frac{(2n)!}{n!(2n)(2n-2)(2n-4)\cdots 4\cdot 2}$$

$$= \frac{\cancel{2^n}(2n)!}{n!\cancel{2^n}n!} = \frac{(2n)!}{n!n!} = \binom{2n}{n}.$$

Medium Technique: Method of Characteristics

- Works for linear, homogeneous, constant coefficient recurrences.
- I.e., things of the form: $a_n f_n + a_{n-1} f_{n-1} + \cdots + a_{n-k} f_{n-k} = 0$

$$a_n, a_{n-1}, ..., a_{n-k} \in \mathbb{C}$$

STEPS:

1. Make the guess $f_n = q^n$ for $q \neq 0.$ Plugging this into the recurrence yields our ***characteristic polynomial*** eqn $P(q) = 0$.
2. Solve for roots $q_1, ..., q_k$ of $P(q)$. Our general solution will be: $f_n = c_1 q_1^n + \cdots c_k q_k^n$.
3. Plug in enough initial conditions to make a $k \times k$ linear system in the unknown c_i.
4. Solve the system.

Medium Technique: Method of Characteristics

Stereotypical EXAMPLE: Fibonacci Numbers

$$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, f_1 = 1.$$

Solution:

- First rewrite the recurrence, verify that it belongs to our class:

$$f_n - f_{n-1} - f_{n-2} = 0$$

- Make a guess: $f_n = q^n, \quad q \neq 0$

$$q^n - q^{n-1} - q^{n-2} = 0$$

$$q^{n-2}(q^2 - q - 1) = 0$$

$$q^2 - q - 1 = 0$$

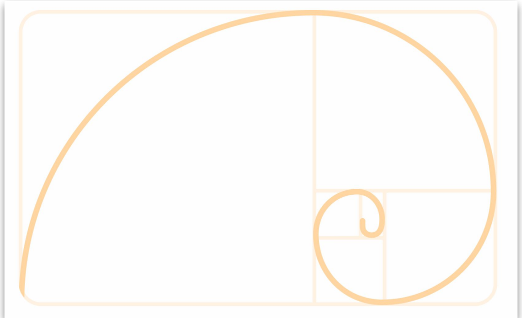
Medium Technique: Method of Characteristics

- Solve this quadratic equation using quadratic formula:

$$\underbrace{q_1 = \frac{1+\sqrt{5}}{2}}_{=\ \phi}, \quad \underbrace{q_2 = \frac{1-\sqrt{5}}{2}}_{=\ 1/\phi}$$

are solutions.

Golden ratio:



- The general solution to the recurrence is of the form:

$$f_n = c_1 q_1^n + c_2 q_2^n.$$

- Now, plug in initial conditions to find c_1, c_2.

$$\begin{array}{l} f_0 = 0 \implies c_1 q_1^0 + c_2 q_2^0 = \\ f_1 = 1 \implies c_1 q_1^1 + c_2 q_2^1 = \end{array} \boxed{\begin{array}{l} c_1 + c_2 = 0 \\ c_1 q_1 + c_2 q_2 = 1. \end{array}}$$

Medium Technique: Method of Characteristics

- Solve the 2x2 linear system in the c’s:

$$\begin{bmatrix} 1 & 1 \\ q_1 & q_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving: $c_1 = \frac{1}{\sqrt{5}}, c_2 = \frac{-1}{\sqrt{5}}.$

Thus, $\forall n,$ $\boxed{f_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\phi^{-n}.}$

Medium Technique: Method of Characteristics

- ***Doesn't always work! We need to be able to factor P(q)
- Will work, with modifications, for the case of:
  - Repeated roots in P(q)
  - Inhomogeneous recurrences, i.e. things like

$$a_n f_n + a_{n-1} f_{n-1} + \cdots + a_{n-k} f_{n-k} = b_n$$
$$a_n, a_{n-1}, ..., a_{n-k} \in \mathbb{C}$$

Advanced Technique: Petkovšek's algorithm

- Works for finding all hypergeometric solutions of a polynomial recurrence:

$$p_n f_n + p_{n-1} f_{n-1} + \cdots + p_{n-k} f_{n-k} = 0$$

where each $p_i = \sum_{m=0}^{d_i} a_m n^m$ is a polynomial in n.

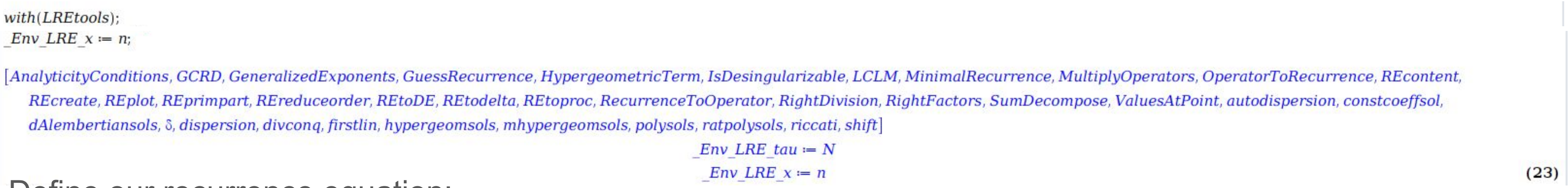
- Will output (a basis for) all hypergeometric solutions if they exist, else output that there are none.
- Relies on the theory of operator algebras.

Advanced Technique: Petkovšek's algorithm

Maple demo. Solve the following recurrence:

$$(n+2)f(n+2) + (n^2 - 2n + 1)f(n+1) - 3n^2 f(n) = 0$$

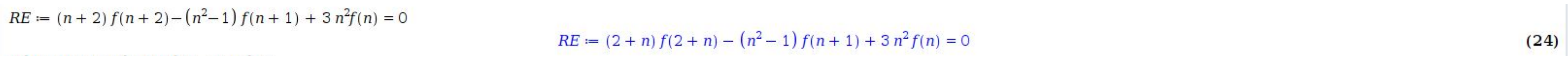
The Maple implementation of Petkovsek’s is in the LREtools package.
Import it, and set independent variable to “n”.



```
with(LREtools);
_Env_LRE_x := n;

[AnalyticityConditions, GCRD, GeneralizedExponents, GuessRecurrence, HypergeometricTerm, IsDesingularizable, LCLM, MinimalRecurrence, MultiplyOperators, OperatorToRecurrence, REcontent,
    REcreate, REplot, REprimpart, REreduceorder, REtoDE, REtodelta, REtoproc, RecurrenceToOperator, RightDivision, RightFactors, SumDecompose, ValuesAtPoint, autodispersion, constcoeffsol,
    dAlembertiansols, δ, dispersion, divconq, firstlin, hypergeomsols, mhypergeomsols, polysols, ratpolysols, riccati, shift]
                                                            _Env_LRE_tau := N
                                                             _Env_LRE_x := n                                                                (23)
```

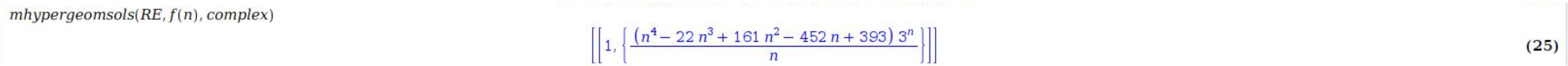
Define our recurrence equation:



$RE := (n+2)\,f(n+2) - (n^2 - 1)\,f(n+1) + 3\,n^2 f(n) = 0$

$$RE := (2+n)\,f(2+n) - (n^2-1)\,f(n+1) + 3\,n^2 f(n) = 0 \qquad (24)$$

Plug into hypergeomsols:



```
mhypergeomsols(RE, f(n), complex)
```

$$\left[\left[1, \left\{\frac{(n^4 - 22\,n^3 + 161\,n^2 - 452\,n + 393)\,3^n}{n}\right\}\right]\right] \qquad (25)$$

V. Conclusion

Thank you! :)