Helly-type Theorems in Geometry

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Have you ever wondered how scheduling surveys like When2Meet work?

Scheduling Example

Each interval represents a participant's single availability.

Finding a time that works for everyone can be tricky, but if every two participants have some overlap, there is surprisingly always a common time for all. **This is explained by an idea in combinatorial geometry called Helly's Theorem.**

Helly’s Theorem in $\mathbb{R}$

Let $\mathcal{S}$ be a collection of **compact and convex sets** in $\mathbb{R}$. If $\mathcal{S}$ is **2-linked** (i.e., every 2 sets have a common point), then the **intersection of all sets in $\mathcal{S}$ is non-empty**:

$$\bigcap_{S \in \mathcal{S}} S \neq \emptyset$$

This is the one-dimensional version of Helly’s theorem.

Helly's Theorem in $\mathbb{R}^d$

Let $\mathcal{C}$ be a collection of **compact and convex sets** in $\mathbb{R}^d$. If every $d+1$ sets in $\mathcal{C}$ have a non-empty intersection, then the intersection of all sets in $\mathcal{C}$ is non-empty:

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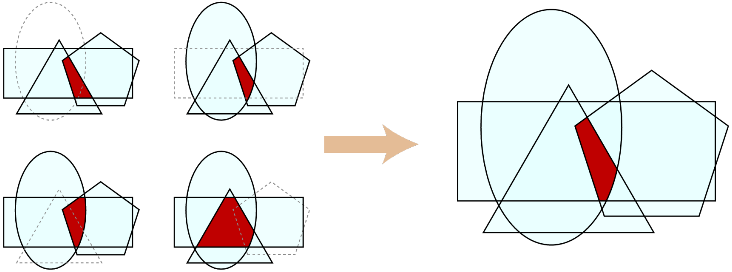


Classic 2D example: every three shapes intersect, so all shapes share a common point.

Convexity

Definition

A set S is said to be convex if for any two points $x, y \subseteq S$ and any scalar $\lambda \subseteq (0, 1)$, the point $\lambda x + (1 - \lambda)y$ also belongs to S.

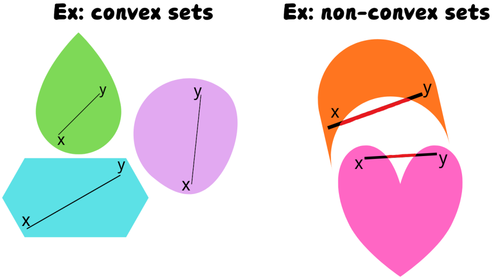


Figure: examples of convex and non-convex sets in R^2

The Role of Compactness

Definition: Compactness in $\mathbb{R}^d$

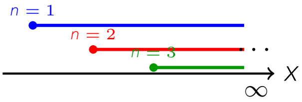
A set $K \subset \mathbb{R}^d$ is **compact** if it is both **closed** and **bounded**.

- **Bounded**: K is contained within a ball of finite radius R ($\exists R > 0$ s.t. $\|x\| \leq R$ for all $x \in K$).
- **Closed**: K contains all its limit points. This means if a sequence $\{x_n\} \subseteq K$ converges to L, then $L \in K$.

Why does it matter for Helly's Theorem?

Consider an **infinite** family of sets $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ in $\mathbb{R}^1$:

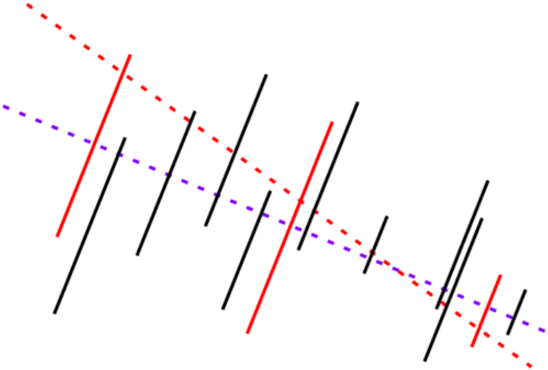
$$F_n = [n, \infty) = \{x \in \mathbb{R} \mid x \geq n\}$$



Santalós Theorem

Santalós Theorem on Transversals

If every **three** of a family of **parallel** segments (finite or infinite) in the R^2 plane has **a transversal** (a line that passes through them), then all segments have a common transversal.



Red: a transversal for any three segments

Purple: the common transversal for all segments

Proof: Santalós Theorem on Transversals

If every **three** of a family of **parallel** segments in the R^2 plane has **a transversal** (a line that passes through them), then all segments have a common transversal.

- Arrange the segments so that they are vertical on the (x, y)-plane with different x-coordinates.
- Each segment have coordinate $\theta = (x_0, y) : y_0 \leq y \leq y_1$. A **transversal** $y = ax + b$ intersects θ if:

$$\mathbf{y_0 \leq ax_0 + b \leq y_1}$$

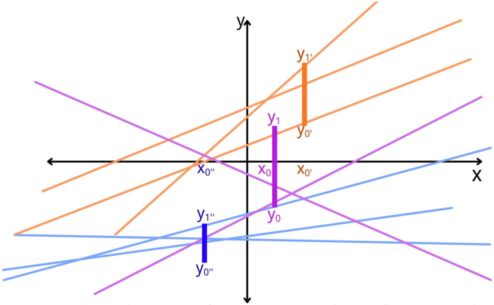


Figure 1: Three (x, y) plane segments with sample transversals.

Proof: Santalós Theorem on Transversals

- Every transversal for a segment form a strip in the **(a, b)** plane bounded by the lines:

$$\mathbf{b} = -\mathbf{x_0 a} + \mathbf{y_0}$$

$$\mathbf{b} = -\mathbf{x_0 a} + \mathbf{y_1}$$ (note: x_0, y_0 and y_1 become coefficients)

- The different segments correspond to different x_0 thus different slopes, which every two strips must intersect and form a compact shape.

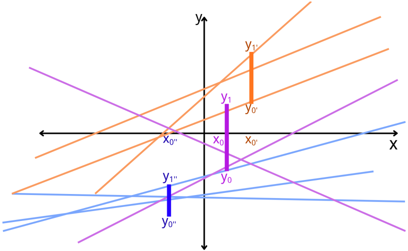


Figure 1: Three (x, y) plane segments with sample transversals.

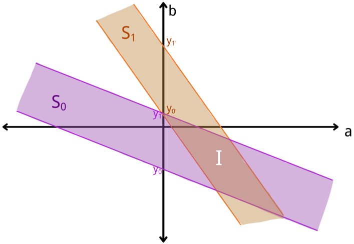
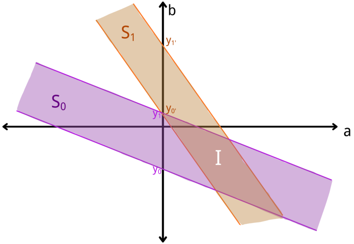


Figure 2: Two transversals from Figure 1 mapped to the (a,b) plane

- In the (a,b) plane, we have a number of convex strips with convex intersections.
- Let $\{S_0, S_1, ...S_n\}$ be our full collection of strips. Since they have different slopes, **the intersection of any two sets, I,** must be a non-empty convex set.
- For any other strip S_j not in the previous intersection, consider $K_j = I \cap S_j$. **Since I and S_j are convex, K_j is convex and $K_j \leq I$.**



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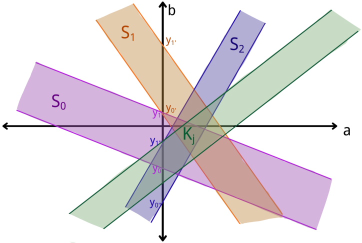

Figure 3

- **Applying Helly's Theorem in R^2:** By hypothesis, any sub-collection of size 3 has a non-empty intersection. Therefore, the collection of all sets has non-empty intersection.

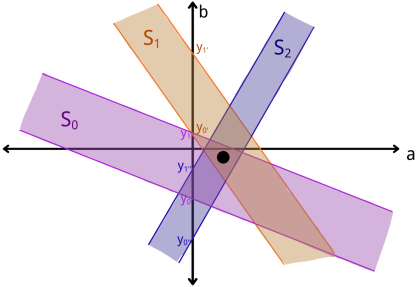

Figure 4: Hellys theorem hypothesis

- **Conclusion:** the collective intersection being non-zero in the (a-b) plane implies at least a common transversal through all segments in the (x-y) plane.

Colorful Helly's Theorem

Colorful Helly's Theorem

Theorem (Colorful Helly). Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be finite families of **convex sets** in $\mathbb{R}^d$ (each thought of as a different color).
If for every **colorful selection**

$$C_1 \in \mathcal{C}_1, \ \ldots, \ C_{d+1} \in \mathcal{C}_{d+1},$$

the intersection

$$\bigcap_{i=1}^{d+1} C_i \neq \emptyset,$$

then there exists an index $j \in \{1, \ldots, d+1\}$ such that

$$\bigcap_{C \in \mathcal{C}_j} C \neq \emptyset.$$

Interpretation: if every “rainbow choice” intersects, then one color class intersects entirely.

Lemma (Matoušek, 2002)

Lemma Let $\mathcal{C}$ be a finite family of **compact, convex sets** in $\mathbb{R}^d$ with

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

Then there exists a subfamily $\mathcal{C}' \subseteq \mathcal{C}$ of size at most d such that

$$\operatorname{lexmin}\left(\bigcap_{C \in \mathcal{C}'} C\right) = \operatorname{lexmin}\left(\bigcap_{C \in \mathcal{C}} C\right)$$

Remark. We use the *lexicographic order* on $\mathbb{R}^d$: $x <_{\text{lex}} y$ if at the first coordinate where they differ, $x_j < y_j$. This ordering lets us define a unique “smallest” point in any compact set, denoted $\operatorname{lexmin}(C)$.

Proof Idea of Colorful Helly's Theorem

- Among all *colorful intersections* $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, choose sets $C_1^*, \ldots, C_{d+1}^*$ that **maximize the lexicographic minimum**:

$$x^* = \mathrm{lexmin}\Big(\bigcap_{i=1}^{d+1} C_i^*\Big)$$

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- By Lemma:

$$\operatorname{lexmin}\Big(\bigcap_{i=1}^{d+1} C_i^*\Big) = \operatorname{lexmin}\Big(\bigcap_{i=1}^{d} C_i^*\Big) = x^*$$

Only *d* sets are needed to determine the lexicographically smallest point of the intersection - so removing one doesn't change it.

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- Key claim: $x^* \in \bigcap_{C \in \mathcal{C}_{d+1}} C$.
- Reason: adding one more set cannot decrease the lexicographic minimum.
- Therefore, one color class intersects, proving the theorem.

Remark: If all color classes are identical, this reduces to the classical Helly Theorem.

Cantor's Intersection Theorem v.s Helly's Theorem

Cantor's Intersection Theorem

Theorem (Compactness-based)

Let $\{C_k\}$ be a sequence of **non-empty, compact** subsets of $\mathbb{R}^d$. If they are **nested** ($C_1 \supseteq C_2 \supseteq \dots$), then:

$$\bigcap_{k=1}^{\infty} C_k \neq \emptyset$$

Key Requirements:

- **Compactness**: Sets must be closed and bounded (prevents ”escaping” to infinity or holes).
- **Nesting**: Each set must be contained in the previous one.
- *Dimension Independent*: Works in any complete metric space.

Visualizing Cantor's Intersection Theorem

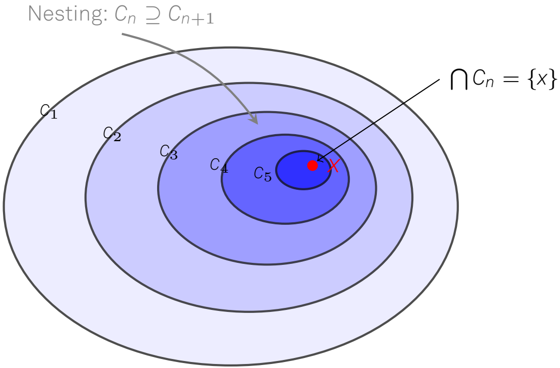
Theorem Statement

Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of non-empty, compact, nested sets in $\mathbb{R}^d$:

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

Then their intersection is non-empty:

$$\bigcap_{n=1}^{\infty} C_n \neq \emptyset$$



Note: If diameters $\to 0$*, the intersection is a single point.*

Think About It

Both theorems guaranty a non-empty intersection, but...
Why can't we simply prove one from the other?

To demonstrate why Helly and Cantor cannot prove each other, we must identify a scenario where one holds true, yet the other one false could be both prerequisites are absent or its logical false

The Missing Bridge: Convex Hulls

Taking the **Convex Hull** (conv(C)) doesn't bridge the gap:

$$\bigcap \mathrm{conv}(C_i) \neq \emptyset \quad \not\Longrightarrow \quad \bigcap C_i \neq \emptyset$$

The intersection of convex hulls **does not** guarantee the intersection of the original sets.

Next, let's look at the counter-examples to see why...

Why Helly Cannot Prove Cantor

The Fundamental Limitation of Helly's Theorem: Helly's Theorem is a *geometric* result that relies strictly on **Convexity.** Cantor's Theorem is a *topological* result that does not.

Helly's ”Weakness”

If the sets are **not convex**, Helly fails. Even if every pair or triple intersects, the total intersection can be empty.

Cantor's ”Strength”

The cantor works for **any** nested compact set (rings, stars, or discrete points), regardless of their shape.

Let's look at a example which use that Helly's ”Weakness” and Cantor's ”Strength”

Example: Why Helly Cannot Prove Cantor

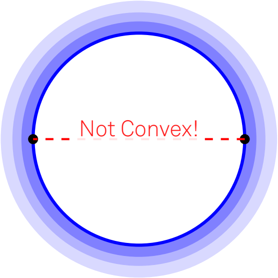
Non-Convex Nested Sets: Why Shape Matters

Cantor: **Success** ✓

- **Nested**: $C_1 \supset C_2 \supset C_3 \dots$
- **Compact**: Closed and bounded.
- **Limit**: $\bigcap C_n =$ Unit Circle $\neq \emptyset$.

Helly: **Failure** ×

- **Non-Convex**: A ring has a hole shows that it's not convex
- **Result**: Convexity test fails because the center is "missing".



$C_n \rightarrow$ Unit Circle

Why Cantor Cannot Prove Helly

The Fundamental Limitation of Cantor's Theorem: Cantor's is a topological result that relies strictly on **Nesting** ($C_1 \supseteq C_2 \supseteq \dots$). Where Helly's does not.

Cantor's "Weakness"

If the sets are **not nested**, Cantor fails immediately. Even if 1,000 sets pairwise intersect, Cantor provides no guarantee.

Helly's "Strength"

Helly works for **any** finite collection of convex sets, even if they are scattered and **none** are contained within others.

A Counter-Example for Cantor (where Helly still works):

- Consider three **solid disks** D_1, D_2, D_3 in $\mathbb{R}^2$ arranged like a Clover/Venn diagram.
- Every two disks intersect, and they are all compact.
- **But they are not nested**: $D_1 \not\supseteq D_2 \not\supseteq D_3$.
- *Result:* Cantor's Theorem remains silent, but Helly's Theorem guaranties a triple intersection.

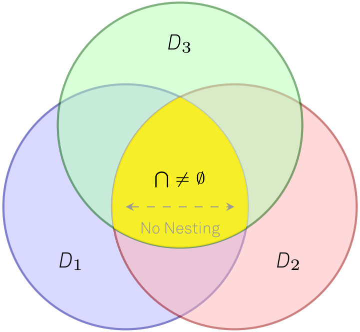
Example: Why Cantor Cannot Prove Helly

Helly: **Success** ✓

- **Convexity**: Each disk D_i is a solid, convex set.
- **Pairwise**: Every two disks overlap.
- **Result**: Geometry guarantees $\bigcap D_i \neq \emptyset$.

Cantor: **Silent** ×

- **Not Nested**: $D_1 \not\supset D_2 \not\supset D_3 \dots$
- **Status**: The "shrinking" machinery is paralyzed.
- **Result**: Cantor provides no information.



Thank You for Listening!

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Feel free to ask any questions!