Analyzing Social Networks: Enumerating Maximal Cliques in c-Closed Graphs

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Introduction & Motivation

- In our DRP, we studied Fox et al.'s paper: "Finding Cliques in Social Networks: A New Distribution-Free Model" (2020) on a new, deterministic model for social networks.
- We focus on the concept of *c-closure* which encapsulates the idea of triadic closure.
- Our goal is to present the core ideas and proof strategies from the paper.

Background on Social Networks

- A graph is a set of vertices connected by edges.
- In a social network, the vertices represent people and the edges represent relationships.
- Social networks exhibit features like:
  - Heavy-tailed degree distributions.
  - High triangle density.
  - Strong *triadic closure*: If two people have many common friends, then they are more likely to be friends.

Past Models for Social Networks

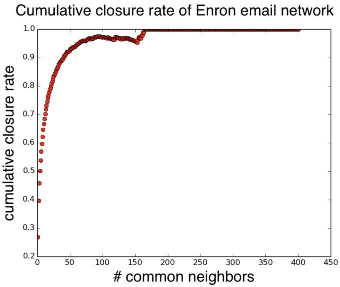
- Many generative models exist
  - preferential attachment
  - copying models
- Fox et al. take a different approach by defining networks through deterministic, combinatorial properties.

Triadic Closure

Definition (Triadic Closure)

If two vertices share a common neighbor, they are likely to be directly connected.

- This observation motivates the formal notion of *c-closure*.
- Empirical data (e.g., Enron emails) confirm these properties.



c-Closed Graphs

Definition (c-Closed Graph)

An undirected graph $G = (V, E)$ is *c-closed* if any two vertices that have at least c common neighbors are adjacent.

- Note: In a c-closed graph, any two nonadjacent vertices have at most $c - 1$ common neighbors.
- Encodes the idea that a high number of common neighbors forces a connection.
- The parameter c can range from 1 to $|V| - 1$.

Example of c-Closed graph

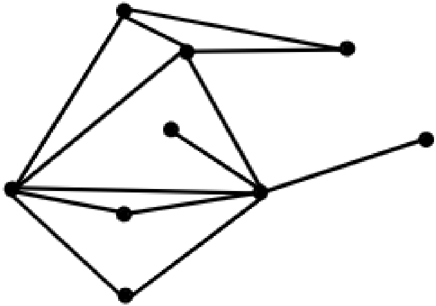


Figure: This graph is 3-closed.

Weakly c-Closed Graphs

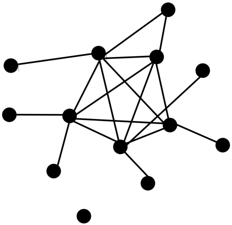
Definition (Weakly c-Closed Graph)

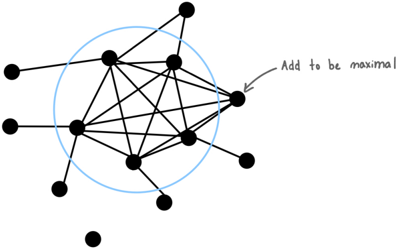
A graph is *weakly c-closed* if there exists an ordering $\{v_1, v_2, \ldots, v_n\}$ such that for every i, v_i is not in any *bad pair* (a non-adjacent pair with $\geq c$ common neighbors) in the subgraph restricted to $\{v_i, v_{i+1}, \ldots, v_n\}$.

- This relaxed version still captures the essence of triadic closure.

Key Concepts

- **Cliques:** A subgraph where every vertex is adjacent to each of the others.
- **Maximal Cliques:** Cliques that cannot be extended by adding any vertex.





Goal

- **FPT (Fixed-Parameter Tractability):** Problems solvable in time $f(c)n^{\alpha}$ (polynomial time) with parameter c (as opposed to "intractable" exponential time).
- Enumerating maximal cliques is a central challenge in social network analysis.

Goal

Improve efficiency of enumerating maximal cliques of a graph

Main Results

The upper and lower bounds of the number of maximal cliques in a c-closed graph from Fox et al are as follows:

Theorem 1.4 (Upper Bound)

In a c-closed graph on n vertices, there are at most

$$\min\{3^{\frac{c-1}{3}} n^2, 4^{\frac{(c+4)(c-1)}{2}} n^{2-2^{1-c}}\}$$

maximal cliques.

Theorem 1.7 (Lower Bound)

For positive integer c, there are c-closed graphs on n vertices and

$$\Omega(c^{-3/2} n^{3/2})$$

maximal cliques.

Proof Strategies

- Upper-bound Strategies
  - **Peeling Process (Argument 1):** Recursively remove a vertex one at a time and classify maximal cliques based on its inclusion.
  - **Case Analysis (Argument 2):** Divide into low and high maximum degree cases ($n^{1/2}$) to optimize bounds.
- Lower-bound Strategies
  - Start with a graph H of girth 5 and replace vertices with cliques to construct a new graph G.
  - Prove it is c-closed with $\Omega(c^{-3/2}n^{3/2})$ maximal cliques

Open Problems

- Tighten the gap between the upper and lower bounds on maximal clique counts.
- Extend these techniques to other NP-hard problems on c-closed graphs.
- Explore additional deterministic models for social networks.
- Find other types of subgraphs besides cliques for c-closed graphs.

Conclusion & Acknowledgments

- Fox et al.'s paper provides a novel deterministic framework for modeling social networks via c-closure.
- Their work establishes key bounds and an FPT algorithm for maximal clique enumeration.
- Our DRP deepened our understanding of graph theory and proof strategies.
- **Acknowledgments:** We thank our mentor, Gabriela Bourla, for her guidance and support.