CO-3 LATTICE-POINT ENUMERATION OF POLYTOPES

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How can we find the area of this polygon?

Example

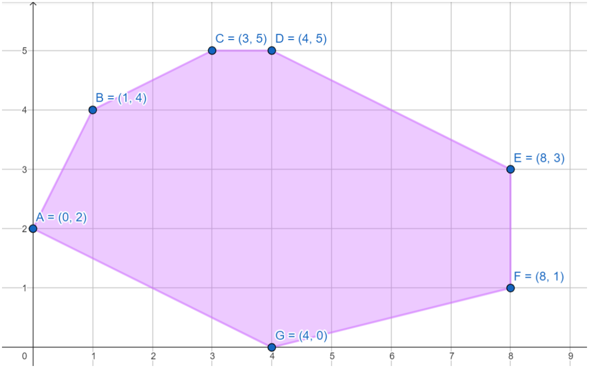


Figure 1: A Lattice Polygon

We'll present a surprising way to find the area of this polygon through discrete methods!

There are two equivalent definitions of polytopes:

1. H-representation: Intersection of Finite Halfspaces

Example

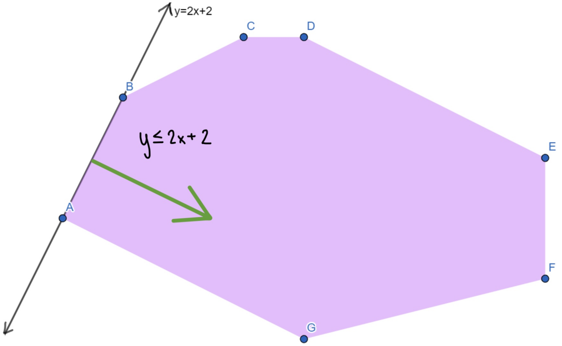


Figure 2: One Halfspace

There are two equivalent definitions of polytopes:

1. H-representation: Intersection of Finite Halfspaces

Example

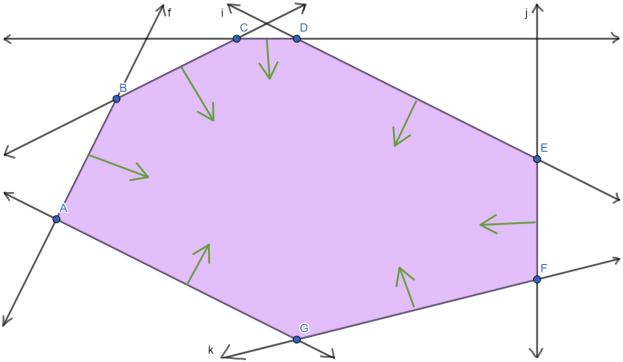


Figure 3: Intersections of Halfspaces

2. V-representation: Convex Hull of a Finite Set of Points (Vectors)

$$P = \text{conv}(v_1, v_2, \ldots, v_k) = \left\{ \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k : \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

Example

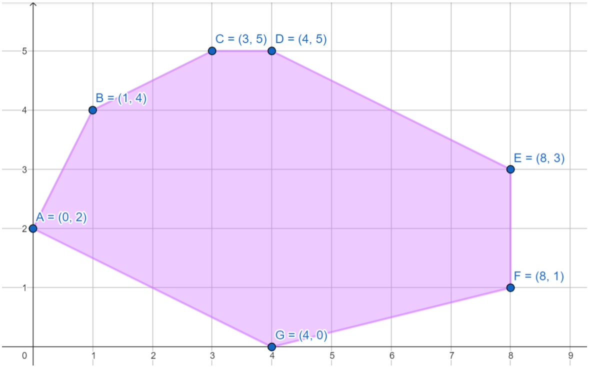


Figure 4: P is represented as the convex hull of A,B,C,D,E,F,G.

In our example, A, B, C, D, E, F, G are the vertices.

Notice: Any line joining two vertices of a polytope is inside its convex hull.

Example

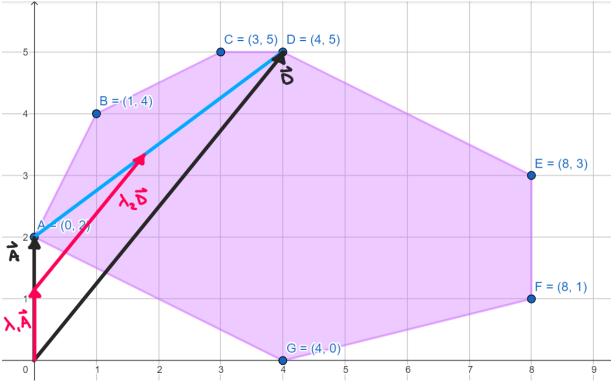


Figure 5: Linear combination of vertices

We want to reduce our polygon into simple parts, via triangulation.

For 2-D Polytopes: Select any vertex and connect every other vertex to the selected vertex.

Example

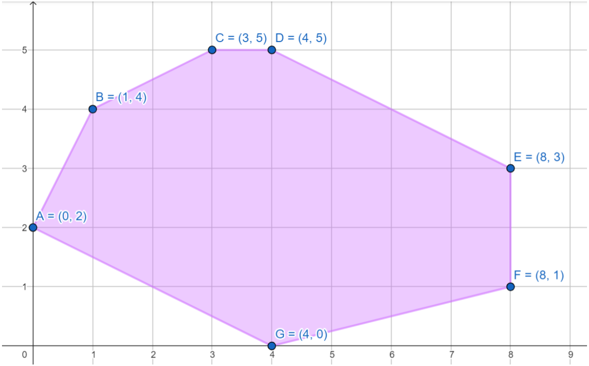


Figure 6: Triangulation of a Polygon

We want to reduce our polygon into simple parts, via triangulation.

For 2-D Polytopes: Select any vertex and connect every other vertex to the selected vertex.

Example

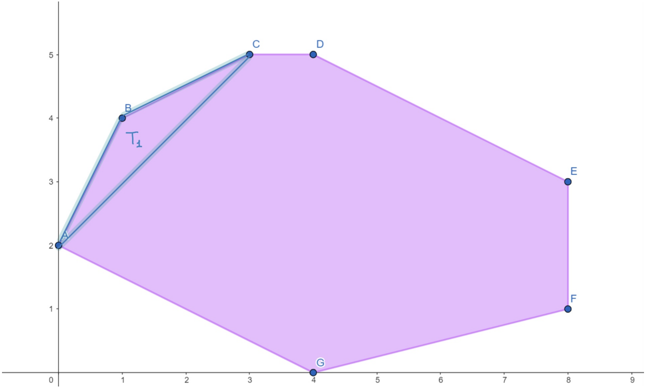


Figure 7: Triangulation of a Polygon

We want to reduce our polygon into simple parts, via triangulation.

For 2-D Polytopes: Select any vertex and connect every other vertex to the selected vertex.

Example

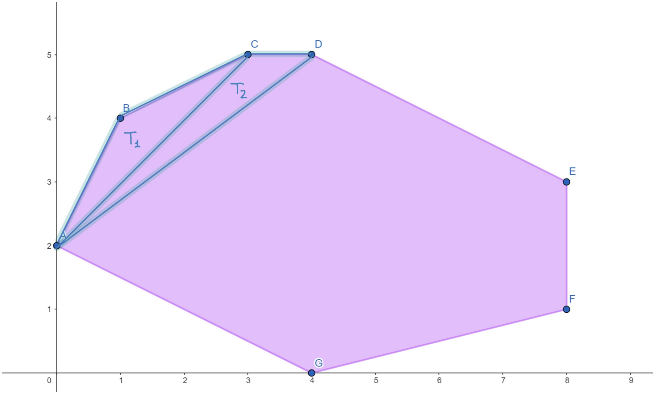


Figure 8: Triangulation of a Polygon

Triangulation

We want to reduce our polygon into simple parts, via triangulation.

For 2-D Polytopes: Select any vertex and connect every other vertex to the selected vertex.

Example

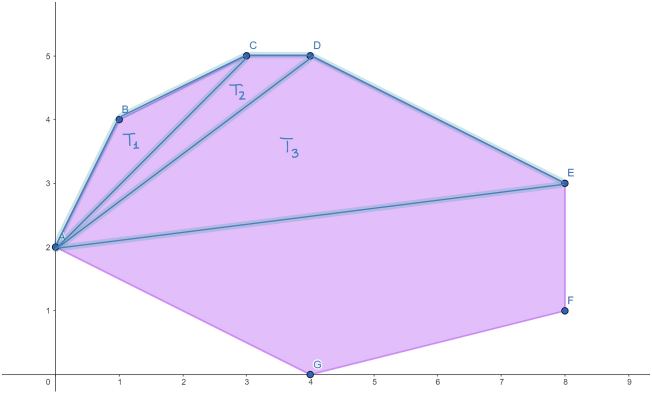


Figure 9: Triangulation of a Polygon

We want to reduce our polygon into simple parts, via triangulation.

For 2-D Polytopes: Select any vertex and connect every other vertex to the selected vertex.

Example

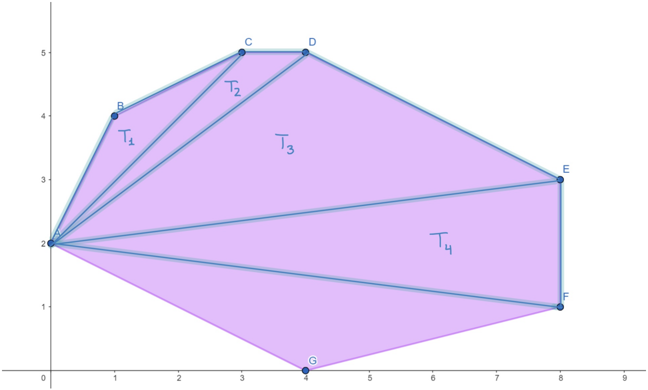


Figure 10: Triangulation of a Polygon

We want to reduce our polygon into simple parts, via triangulation.

For 2-D Polytopes: Select any vertex and connect every other vertex to the selected vertex.

Example

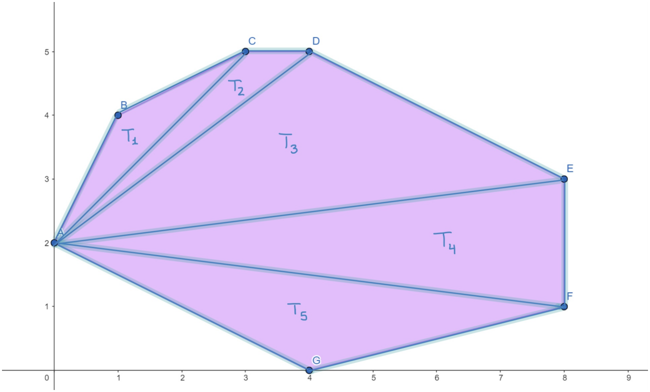


Figure 11: Triangulation of a Polygon

Theorem (Pick, 1899)

Let P be a lattice polygon. Denote by I the number of lattice points in the interior of P, B the number of lattice points in the boundary of P and A_P the area of the polygon. Then

$$A_P = I + \frac{B}{2} - 1.$$

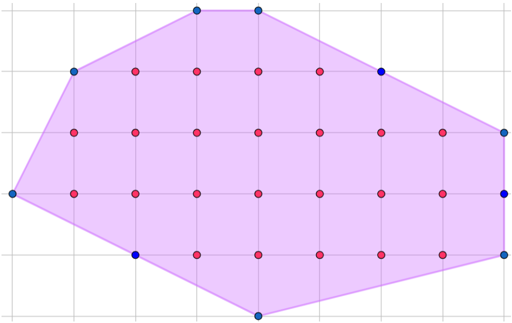


Figure 12: Lattice Points of a Polytope

Sketch of Pick's Theorem Proof for Convex Polygons

1. Pick's Theorem holds for rectangles and right-triangles (that have sides parallel to the axes).

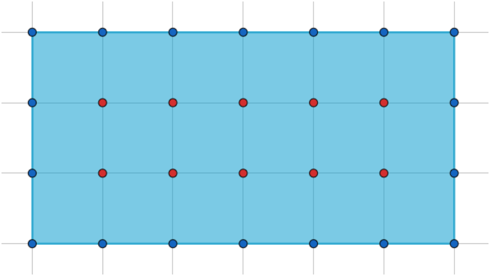
2. Pick's Theorem is "additive" (and "subtractive")

3. Since every lattice triangle is a "sum/difference" of rectangles and rectangle-triangles, Pick's Theorem holds for all lattice triangles.

4. Since convex lattice polygons can be triangulated into lattice triangles and Pick's theorem is "additive", we conclude that Pick's theorem holds for all convex polygons.

Pick's Theorem Proof: Rectangle

We begin by showing that Pick's Theorem holds for rectangles that have sides parallel to the axes.



Suppose that ℓ is the length of the rectangle and w is the width. We have that

$$I = (\ell - 1)(w - 1) \quad \text{and} \quad B = 2(\ell + w)$$

Thus,

$$\begin{aligned} I + \frac{B}{2} - 1 &= (\ell - 1)(w - 1) + \frac{1}{2} \cdot 2(\ell + w) - 1 \\ &= \ell w - \ell - w + 1 + \ell + w - 1 = \ell w = A_{rect}. \end{aligned}$$

Next, we will prove that Pick's Theorem holds for right triangles.

Example

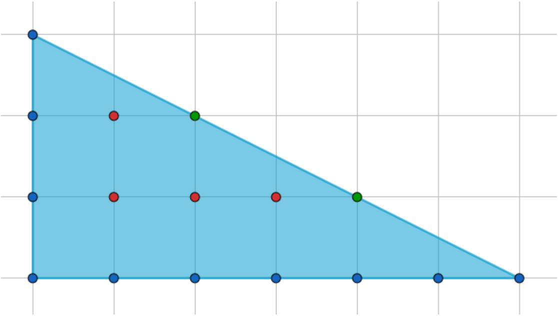


Figure 13: Right Triangle

We will separate the boundary points into two groups:

$$B_p + B_h = B$$

Pick's Theorem Proof: Right Triangle

Comparing to the rectangle that the triangle is embedded in:

Example

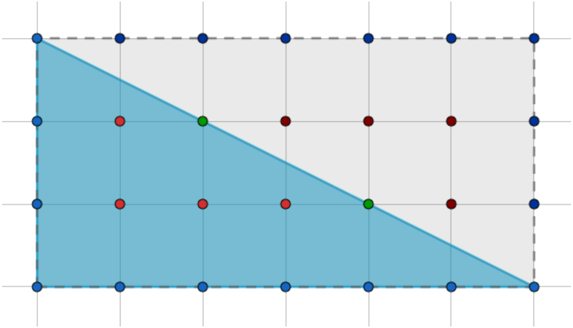


Figure 14: Right Triangle with Rectangle

Notice that

$$2B_p - 2 = B_{rect} \quad \text{and} \quad 2I + B_h = I_{rect}$$

Pick's Theorem Proof: Right Triangle

Rearranging, we get

$$\frac{B_p}{2} = \frac{B_{rect}}{4} + \frac{1}{2} \quad \text{and} \quad I + \frac{B_h}{2} = \frac{I_{rect}}{2}$$

Thus,

$$\begin{aligned} I + \frac{B}{2} - 1 &= I + \frac{B_h}{2} + \frac{B_p}{2} - 1 \\ &= \frac{I_{rect}}{2} + \frac{B_{rect}}{4} + \frac{1}{2} \\ &= \frac{1}{2}\left(I_{rect} + \frac{B_{rect}}{2} + 1\right) \\ &= \frac{1}{2}A_{rect} \\ &= A_{triangle}. \end{aligned}$$

Next, we show that Pick's Theorem has an additive character:

Proposition ("Additivity")

Assume that polygons P_1 and P_2 satisfy Pick's Theorem, and their intersection is a polygonal curve. Then, the polygon $P = P_1 \cup P_2$ also satisfies Pick's Theorem.

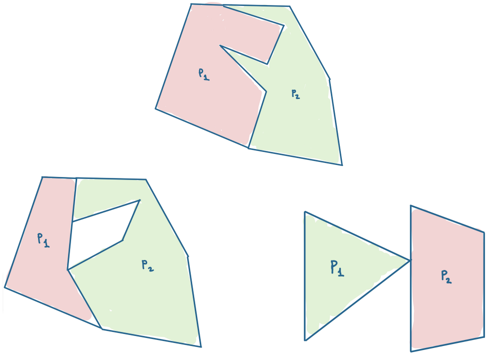


Figure 15: Additivity and Non-Additivity Examples

Pick's Theorem Proof: Additive Condition

Let L be the number of lattice points on the edge common to P_1 and P_2.

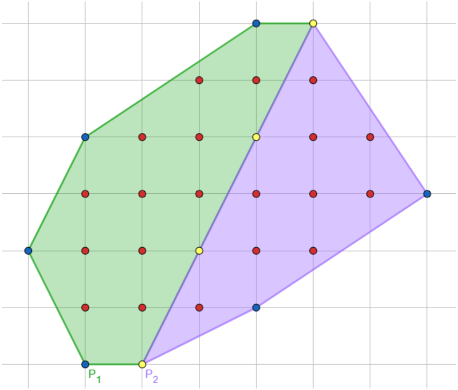


Figure 16: In this example, P_1 is green and P_2 is purple

Notice that

$$I = I_1 + I_2 + L - 2 \quad \text{and} \quad B = B_1 + B_2 - 2L + 2.$$

Pick's Theorem Proof: Additive Condition

Thus,

$$\begin{aligned} I + \frac{B}{2} - 1 &= I_1 + I_2 + L - 2 + \frac{B_1 + B_2 - 2L + 2}{2} - 1 \\ &= I_1 + I_2 + L - 2 + \frac{B_1}{2} + \frac{B_2}{2} - L + 1 - 1 \\ &= I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1 \\ &= A_1 + A_2 \\ &= A. \end{aligned}$$

We can also prove a similar "subtractive" property of Pick's Theorem. That is, if we assume that the intersection of P_1 and P_2 is a polygonal curve, and we assume that P and P_1 both satisfy Pick's Theorem, then P_2 also satisfies Pick's Theorem.

Pick's Theorem Proof: Lattice Triangles

Every lattice triangle is the "sum" and/or "difference" of rectangles and right triangles.

Example

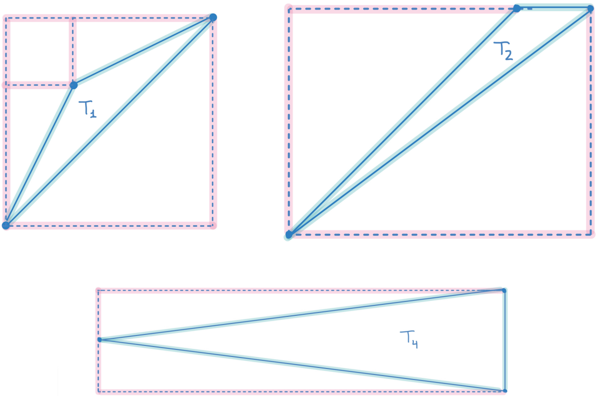


Figure 17: Lattice triangles as the sum and/or difference of rectangles and right triangles

Pick's Theorem Proof: Summary

In short, by additivity:

Lattice rectangles and right triangles satisfy Pick's Theorem

$\Longrightarrow$ All lattice triangles satisfy Pick's Theorem, since they are the sum and/or difference of rectangles and right triangles with sides parallel to the axes.

$\Longrightarrow$ All convex polygons satisfy Pick's Theorem, since we can triangulate any convex polygon into lattice triangles.

Example

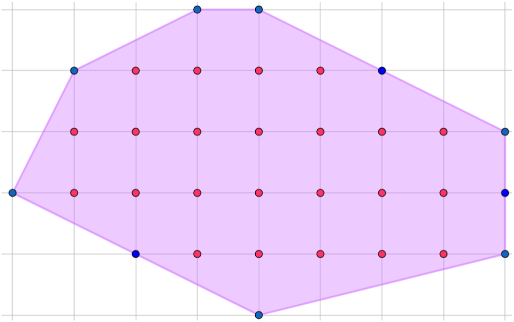


Figure 18: Pick's Theorem Example

In this example: $I = 23$, $B = 10$, so Pick's Theorem says

$$A_P = I + \frac{B}{2} - 1 = 23 + \frac{10}{2} - 1 = 27.$$

Pick's Theorem for Convex Polygons

An example for you to try!

Example

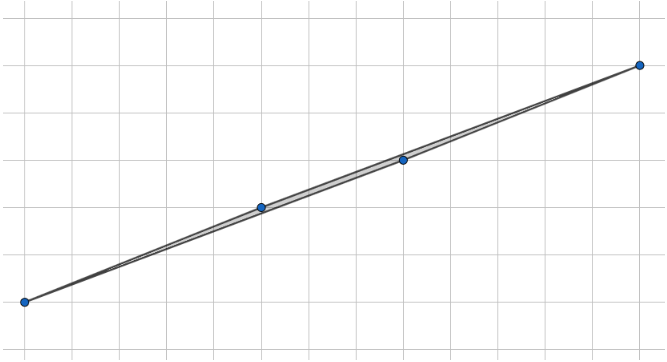


Figure 19: Suspiciously Thin Parallelogram

Pick's Theorem:

$$A_P = I + \frac{B}{2} - 1$$

An example for you to try!

Example

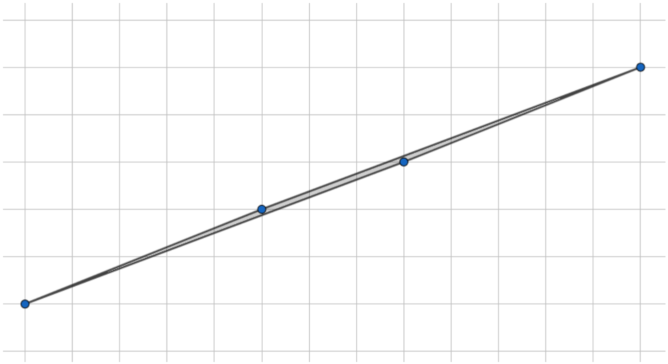


Figure 20: Suspiciously Thin Parallelogram

In this example: $I = 0$, $B = 4$, so Pick's Theorem says

$$A_P = I + \frac{B}{2} - 1 = 0 + \frac{4}{2} - 1 = 1.$$

In fact, this parallelogram is behind the famous missing square optical illusion!

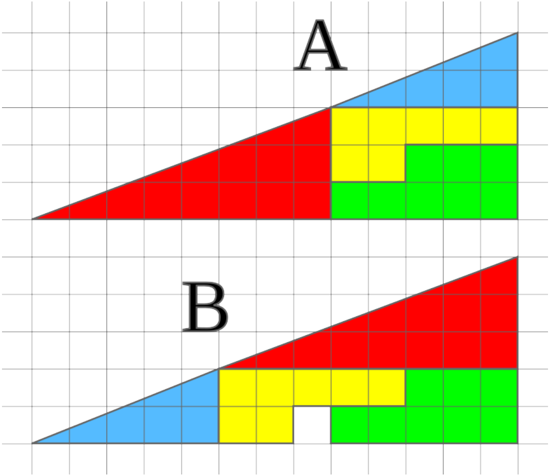


Figure 21: Missing Square Illusion

Image taken from `https://en.wikipedia.org/wiki/Missing_square_puzzle`.

Does Pick's Theorem only hold true for lattice polygons with interior angles greater than π? Let's look at an example!

Example

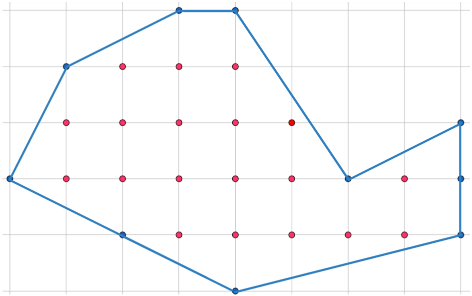


Figure 22: Polytope with an interior angle greater than π

Can we triangulate this?

We first choose one vertex to which we connect all other vertices, but we notice that we cannot reach two vertices while staying within the bounds of the polygon.

Example

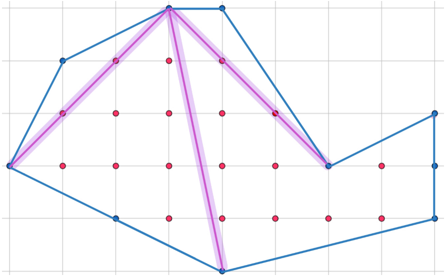


Figure 23: First Triangulation of polygon with an interior angle greater than π

So, how do we further triangulate this?

We choose another vertex to which we connect remaining vertices!

Example

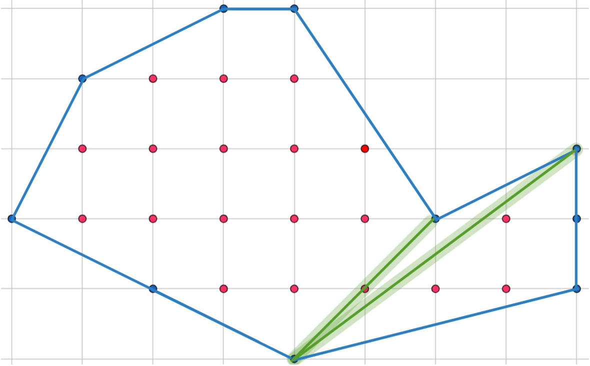


Figure 24: Second Triangulation of polygon with an interior angle greater than π

We now have a triangulated polygon, which are again each the "sum" and/or "difference" of rectangles and right triangles.

Example

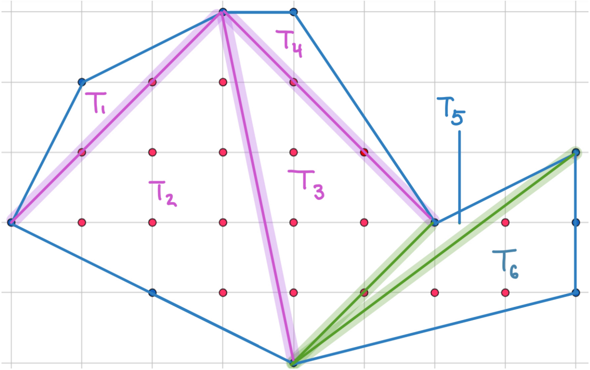


Figure 25: Second Triangulation of polygon with an interior angle greater than π

Example

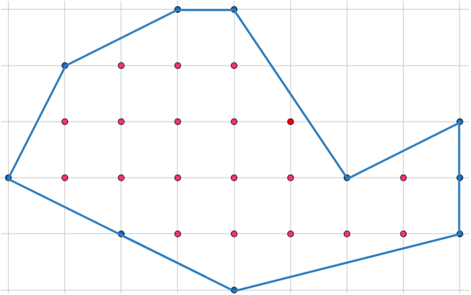


Figure 26: Polygon with interior angles greater than π

In this example: $I = 19$, $B = 10$, so Pick's Theorem says

$$A_P = I + \frac{B}{2} - 1 = 19 + \frac{10}{2} - 1 = 23$$

Further Investigation of Pick's Theorem

There is much more to investigate!

More Generalizations of Pick's Theorem:

- Does Pick's Theorem hold for 2-D objects with one or more holes?
- Does Pick's Theorem hold for other non-convex subsets of the plane?
- Are there similar theorems for higher dimensions? (Search: Reeve's Theorem)

Other proofs of Pick's Theorem:

- via Ehrhart Theory (algebraic)
- via Euler Characteristic (graph theoretic)

Thank you!

References I

[1] M. Beck and S. Robins. *Computing the Continuous Discretely: Integer-point Enumeration in Polyhedra*. Undergraduate Texts in Mathematics. Springer New York, 2007. ISBN: 9780387461120. URL: https://books.google.ca/books?id=laweX9bzDM8C.

[2] Georg Pick. "Geometrisches zur Zahlenlehre". In: *Sitzungsberichte des Deutschen Naturwissenschaftlich-Medicinischen Vereines für Böhmen Lotos in Prag* 19 (1899), pp. 311–319.

[3] G.M. Ziegler. *Lectures on Polytopes*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9780387943657. URL: https://books.google.ca/books?id=xd25TXSSUcgC.