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Arithmetic Functions and their Distributions

Under the mentorship of Sourabh Das

F23 Directed Reading Program
Women in Mathematics

Outline

1. Distribution of arithmetic functions and why we study them?
   - Introduction to arithmetic functions
   - Types of Arithmetic Functions
   - Distributions that we studied
   - Properties of Multiplicative Functions

2. Tools required to study the average distributions
   - Tool 1 - The Big-O
   - Tool 2 - Euler's summation formula
   - Distribution of the Divisor function
   - Tool 3 - Möbius Inversion Formula
   - Distribution of the Euler ϕ-function

Aim of the project

Arithmetic Function

A function $f : \mathbb{N} \to \mathbb{R}$ or $\mathbb{C}$ which aims to study the divisibility properties of integers and/or distribution of prime numbers.

Example: $d(n)$ counts the number of divisors of n. So, $d(6) = 4$, $d(100) = 9$.

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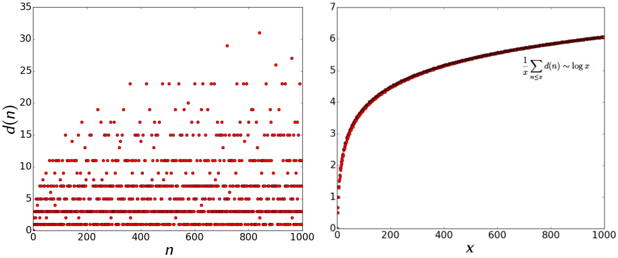

Figure: $\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} d(n) \approx \log x$

To understand how functions behave on average and to gather key insights into their mathematical structures we study their **average distribution**.

Types of Arithmetic Functions

Multiplicative Function

An arithmetic function f satisfying $f(mn) = f(m)f(n)$ whenever gcd$(m, n) = 1$.

Example: (i) $d(n)$, (ii) Euler's totient function, $\phi(n)$ which counts the number of positive integers co-primes to n and $\leq n$, (iii) Möbius function, $\mu(n) = (-1)^r$ if n is square-free with r prime factors.

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Completely Multiplicative Function

An arithmetic function f satisfying $f(mn) = f(m)f(n), \forall m, n \in \mathbb{N}$.

Example: $f(n) = 1$ and $f(n) = n$. Distributions well-known:

$$\sum_{n \leq N} 1 = N, \quad \sum_{n \leq N} n = \frac{N(N+1)}{2}.$$

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Non-Multiplicative Function

Arithmetic functions not satisfying the multiplicative property.

Example: Von Mangoldt function, $\Lambda(n) = \log p$ if $n = p^m, \ m \geq 1$ and 0 otherwise .

Distribution of Multiplicative Functions

We focus our study on the distribution of multiplicative functions, with a particular emphasis on $d(n)$ (Divisor function) and $\phi(n)$ (Euler's totient function).

Determining the average distributions of $\mu(n)$ (Möbius function) and $\Lambda(n)$ (Von Mangoldt function) poses significantly greater challenges. Specifically:

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \mu(n) = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \Lambda(n) = 1.$$

These findings hold equivalent significance to the prime number theorem:

$$\lim_{x\to\infty} \frac{\#\{\text{primes } \leq x\}}{x/\log x} = 1.$$

However, proving the prime number theorem involves methods that were outside the content of the project.

Properties of Multiplicative Functions

Theorem

Let f be a multiplicative function. Then

- $f(1) = 1$
- f is completely multiplicative if and only if $f(p^n) = f(p)^n$ for all primes p and all integers $n \geq 1$.

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Dirichlet Convolution Theorem

If f and g are multiplicative functions, then their Dirichlet convolution $f * g$ given by $(f * g)(n) = \sum_{e|n} f(e) \cdot g\left(\frac{n}{e}\right)$ is also multiplicative.

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Proof: Let m and n be coprime integers. Then

$$\begin{aligned}(f * g)(mn) &= \sum_{e|mn} f(e) \cdot g\left(\frac{mn}{e}\right) = \sum_{e_1|m} \sum_{e_2|n} f(e_1 \cdot e_2) \cdot g\left(\frac{mn}{e_1 \cdot e_2}\right) \\ &= \left(\sum_{e_1|m} f(e_1) \cdot g\left(\frac{m}{e_1}\right)\right) \cdot \left(\sum_{e_2|n} f(e_2) \cdot g\left(\frac{n}{e_2}\right)\right) \\ &= (f * g)(m) \cdot (f * g)(n).\end{aligned}$$

The Big-O (Tool 1)

Let $g(x) > 0$ for all $x \geq a$, we say

$$\mathbf{f(x) = O(g(x))}$$

if there exists $M > 0$ and $x_0 \geq a$ such that $|f(x)| \leq Mg(x)$ for all $x \geq x_0$.

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If $f_1(x) = O(g_1(x)), f_2(x) = O(g_2(x))$, then

$$(f_1 + f_2)(x) = O(g_1(x) + g_2(x)) = O(\max\{g_1(x), g_2(x)\}),$$

$$(f_1 f_2)(x) = O((g_1 g_2)(x)).$$

Example: $x^2 + 2x + 1 = O(x^2)$, $\log x = O(x^{1/1000})$.

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Asymptotic

We say f is asymptotic to g (or write $f(x) \sim g(x)$) if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Example: $x \sim x + 1$, $e^x + x^{100} + \log x \sim e^x$.

Euler's summation formula (Tool 2)

If f has a continuous derivative f' on the interval $[1, x]$, then

$$\sum_{1<n\leq x} f(n) = \int_1^x f(t)dt + \int_1^x (t-[t])f'(t)dt + f(1)([x]-x),$$

where $[t]$ denotes the greatest integer $\leq t$.

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Applications (Completely multiplicative f)

(a) $\sum_{n\leq x} \frac{1}{n} = \log x + C + O(\frac{1}{x})$,

(b) $\sum_{n\leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ if $\alpha \geq 0$.

Euler's constant $C = \lim_{N\to\infty}(\sum_{n=1}^N \frac{1}{n} - \log N)$.

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Proof of (a):

$$\sum_{1\leq n\leq x} \frac{1}{n} = 1 + \sum_{1<n\leq x} \frac{1}{n} = 1 + \int_1^x \frac{1}{t}dt - \int_1^x \frac{t-[t]}{t^2}dt - \frac{x-[x]}{x}$$

$$= \log x + \underbrace{1 - \int_1^\infty \frac{t-[t]}{t^2}dt}_{C} + \underbrace{\int_x^\infty \frac{t-[t]}{t^2}dt}_{<1/x} + O(\frac{1}{x})$$

Distribution of the Divisor function

Recall $d(n) = \sum_{e|n} 1$. We establish

$$\frac{1}{x}\sum_{n\leq x} d(n) \sim \log x.$$

Specifically, $\frac{1}{x}\sum_{n\leq x} d(n) = \log x + (2C+1) + O(1/\sqrt{x})$.

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Proof:

$$\sum_{n\leq x} d(n) = \sum_{n\leq x}\sum_{e|n} 1 = \sum_{e\leq x}\sum_{n\leq x/e} 1 \overset{(b)}{=} \sum_{e\leq x}\left(\frac{x}{e} + O(1)\right) \overset{(a)}{=} x\log x + O(x).$$

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Generalized Divisor function

Let $n \ge 1$ be an integer, $\sigma_\alpha(n) = \sum_{e|n} e^\alpha$ for any real α

$$\frac{1}{x}\sum_{n\le x}\sigma_\alpha(n) \sim \begin{cases} \frac{\zeta(\alpha+1)}{\alpha+1}x^\alpha & \text{if } \alpha > 0, \\ \zeta(-\alpha+1) & \text{if } \alpha < 0, \end{cases}$$

where $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ is the Riemann zeta function defined on $s > 1$.

Möbius Inversion Formula (Tool 3) and its application

Recall the Möbius function defined as $\mu(1) = 1$, and if $n > 1, n = \prod_{k=1}^{n} p_k^{a_k}$.

$$\mu(n) = \begin{cases} (-1)^n & \text{if } a_1 = ... = a_n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Möbius Inversion Formula

Let f and g be two arithmetic functions. Then

$$f(n) = \sum_{e|n} g(e) \quad \Longleftrightarrow \quad g(n) = \sum_{e|n} \mu(e) f\left(\frac{n}{e}\right).$$

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$$f(n) = \sum_{e|n} g(e) \iff g(n) = \sum_{e|n} \mu(e) f\left(\frac{n}{e}\right).$$

For $n \geq 1$, $\phi(n)$ is defined as

$$\phi(n) = \sum_{\substack{1 \leq k \leq n \\ \text{gcd}(n,k)=1}} 1.$$

Thus by Möbius Inversion Formula

$$\underbrace{\sum_{e|n} \phi(e) = n}_{\text{Easy to prove}} \iff \phi(n) = \sum_{e|n} \mu(e) \frac{n}{e}.$$

Distribution of Euler ϕ-function

Recall

$$\text{(a)} \sum_{n \le x} \frac{1}{n} = \log x + O(1), \quad \text{and} \quad \text{(b)} \sum_{n \le x} n = \frac{x^2}{2} + O(x).$$

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For $x > 1$ we have

$$\frac{1}{x}\sum_{n\leq x} \phi(n) = \frac{3}{\pi^2}x + O(\log x).$$

Distribution of Euler ϕ-function

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$$\frac{1}{x}\sum_{n\le x}\phi(n)=\frac{3}{\pi^2}x+O(\log x).$$

Proof: One can show that $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}=\frac{6}{\pi^2}$. Then

$$\begin{aligned}\sum_{n\le x}\phi(n)&=\sum_{n\le x}\sum_{e\mid n}\mu(e)\frac{n}{e}=\sum_{\substack{q,e\\ qe\le x}}\mu(e)q\\ &=\sum_{e\le x}\mu(e)\sum_{q\le x/e}q\\ &\overset{(b)}{=}\frac{1}{2}x^2\sum_{e\le x}\frac{\mu(e)}{e^2}+O(x\sum_{e\le x}\frac{1}{e})\\ &\overset{(a)}{=}\frac{3}{\pi^2}x^2+O(x\log x).\end{aligned}$$


THANK
YOU!