Mermin-Peres Magic Square Game

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Magic Square Game

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

Rules of the Game:

- $x_i \in \{0, 1\}$
- Row sums are even.
- Column sums are odd.
- Referee assigns Alice (A) a row and Bob (B) a column. A and B return triplets of bits that follow the sum rules and also agree on the intersecting square.
- Alice and Bob can decide on a strategy beforehand but cannot communicate during the game.

Magic Square Game

Can A and B come up with a winning strategy?

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

- No.
- Exercise: The best they can do is win 8/9 of the time.

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Basic Quantum Information

- Classical bits are elements of $\{0, 1\}$.
- Quantum bits are unit vectors $|\psi\rangle \in \text{span}\{|0\rangle, |1\rangle\}$.
- Quantum Measurement: "Forcing" a quantum bit to become classical.
  - Can assume this is a unitary acting on $|\psi\rangle$. (Think of this as rotating $|\psi\rangle$.)
  - If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $U = Id$, then $|\psi\rangle$ has a $|\alpha|^2$ chance of becoming $|0\rangle$ and a $|\beta|^2$ chance of becoming $|1\rangle$.

Quantum Entanglement

- A quantum bit lives in $\mathbb{C}^2$. A quantum *state* lives in $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$.
- In the Magic Square Game, our quantum state will live in

$$|\psi\rangle \in \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2}_{\mathcal{H}_A} \otimes \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2}_{\mathcal{H}_B}.$$

A quantum state is *separable* if it can be written as a simple tensor in $\mathcal{H}_A \otimes \mathcal{H}_B$. Otherwise it is *entangled*.

  - Note: $|0101\rangle$ is shorthand for $|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$.

Quantum Strategy for Magic Square Game

- Alice and Bob fix a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\psi\rangle = \frac{1}{2}|0011\rangle - \frac{1}{2}|0110\rangle - \frac{1}{2}|1001\rangle + \frac{1}{2}|1100\rangle$$

.

  Alice can only measure the part of the quantum state in $\mathcal{H}_A$, and similarly for Bob.
- Alice and Bob come up with measurements that they will implement *depending on what row/column the referee assigns.*
- The measurements on their respective Hilbert spaces:

$$A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix} \quad A_2 = \frac{1}{2}\begin{pmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{pmatrix} \quad A_3 = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

$$B_1 = \frac{1}{2}\begin{pmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix} \quad B_2 = \frac{1}{2}\begin{pmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{pmatrix} \quad B_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Example

Suppose the referee gives Alice row 2 and Bob column 3. Alice and Bob "rotate" $|\psi\rangle$ by $A_2 \otimes B_3$

$$A_2 \otimes B_3|\psi\rangle = \frac{1}{2\sqrt{2}}\Big[|0000\rangle - |0010\rangle - |0101\rangle + |0111\rangle$$
$$|1001\rangle + |1011\rangle - |1100\rangle - |1110\rangle\Big]$$

and then the resulting quantum state "snaps" into one of these basis vectors with equal probability (since the coefficients all have the same modulus).

Example (cont.)

$$A_2 \otimes B_3|\psi\rangle = \frac{1}{2\sqrt{2}}\Big[|0000\rangle - |0010\rangle - |0101\rangle + |0111\rangle$$
$$|1001\rangle + |1011\rangle - |1100\rangle - |1110\rangle\Big]$$

- Suppose $A_2 \otimes B_3|\psi\rangle \mapsto |1011\rangle$.
- Alice: first two row entries are 10, so her last entry must be 1 (to sum to an even number).
- Bob: first two column entries are 11, so his last entry must be 1 (to sum to an odd number).
- They win!

		1
1	0	1
		1

Summary

- This works for *every* $A_j \otimes B_k$, and *every* possible measurement outcome.
- Thus Alice and Bob win with *100% probability* using the quantum strategy.

Further Exploration

The Magic Square Game is an example of a nonlocal game in which there exists a quantum strategy that can beat every classical one.

There are even larger classes of games in which we relax these assumptions – what is the best possible strategy over all finite dimensional $\mathcal{H}_A$ and $\mathcal{H}_B$? What if we allow $\mathcal{H}_A$ and $\mathcal{H}_B$ to be infinite dimensional? What if $\mathcal{H}_A = \mathcal{H}_B$?

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Thank you!