Understanding the Poincaré Conjecture

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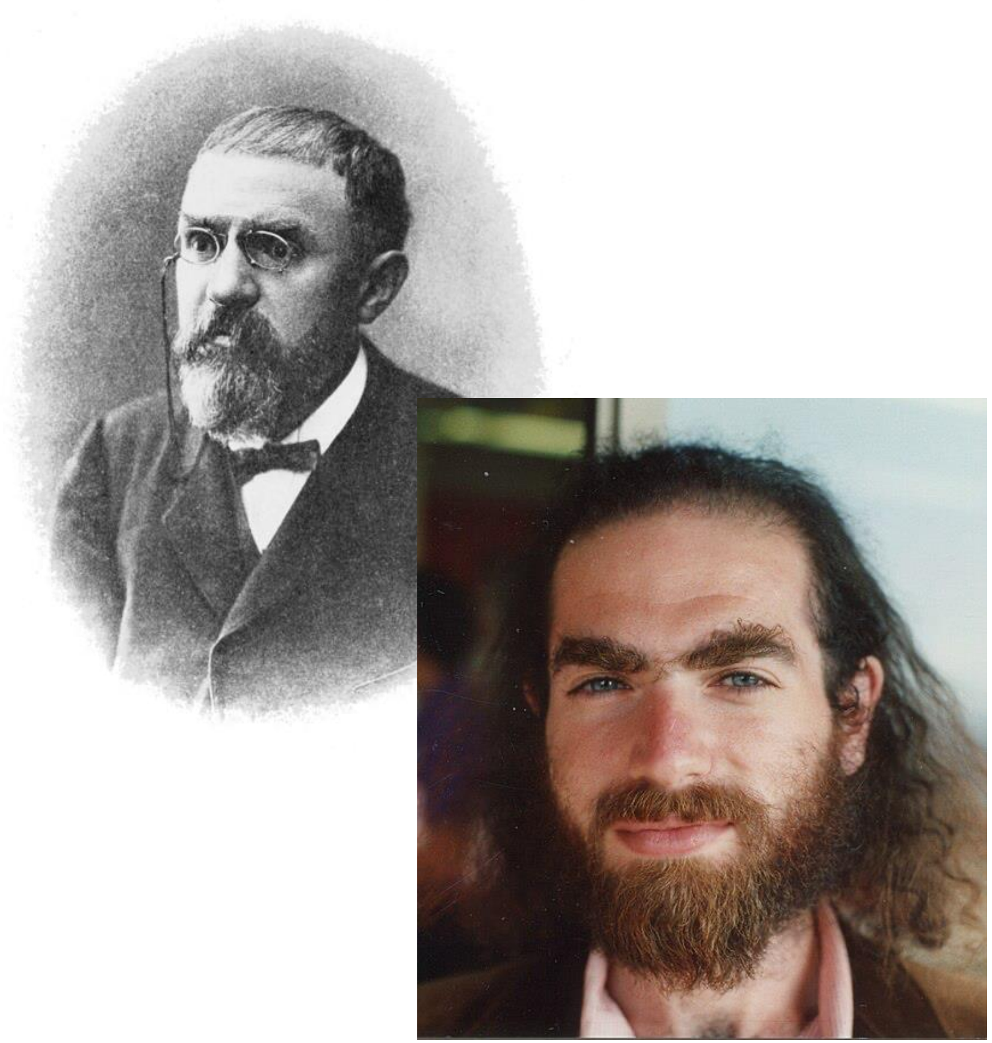
Background of Poincaré Conjecture



- Geometric topology
- Proposed in 1904
- By Henri Poincaré
- The nature of three-dimensional space
- Proof in 2002
- By Grigori Perelman

Fundamental question

If a three-dimensional space has no holes, must it be a three-dimensional sphere?

The formal statement

Poincaré conjecture

Every three-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the three-dimensional sphere.

Understanding the Poincaré Conjecture

Poincaré conjecture

Every three-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the three-dimensional sphere.

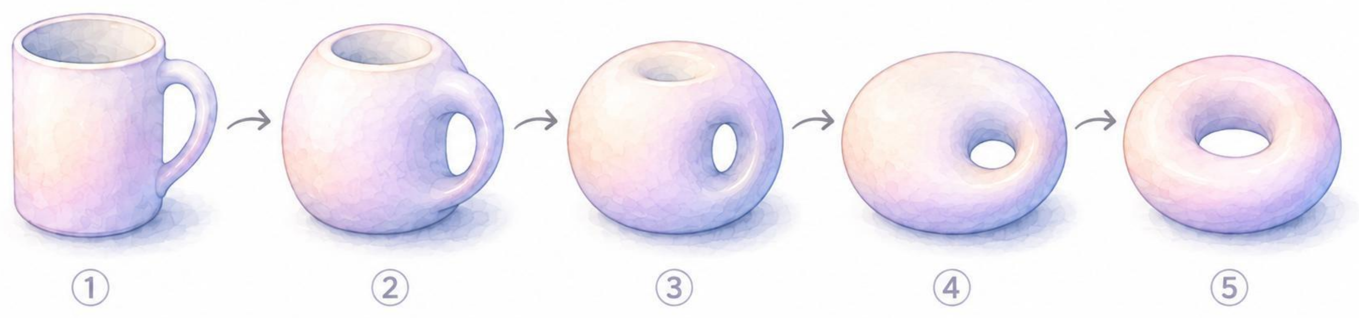
- What is a topological manifold? And how it could be closed and connected?
- What is the fundamental group, and what does "trivial" mean?
- What does it mean for two spaces to be homeomorphic?
- What is a 3-dimensional sphere?

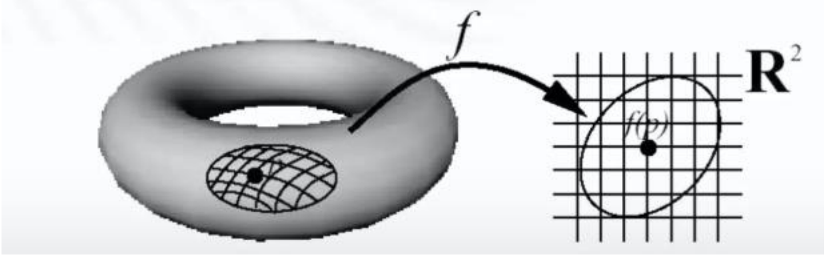
Spheres in different dimensions

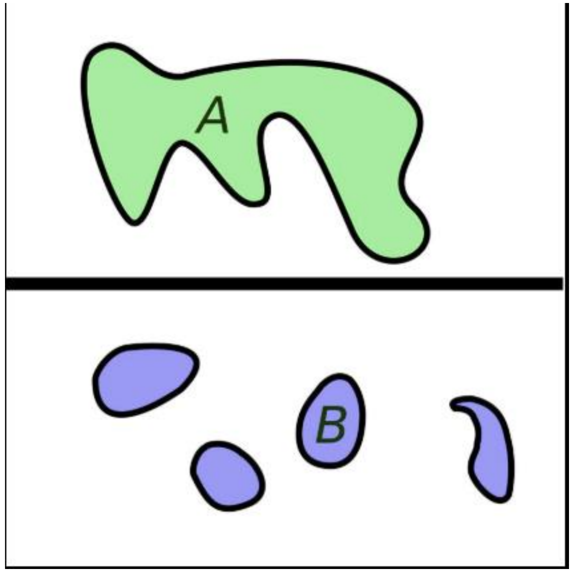
Dimension	**Object**	**Description**
0D	Point	Single dot
1D	Circle (S^1)	All points at distance r from a center in 2D space (circle)
2D	Sphere (S^2)	All points at distance r from a center in 3D space (surface of a ball)
3D	3-Sphere (S^3)	All points at distance r from a center in 4D space

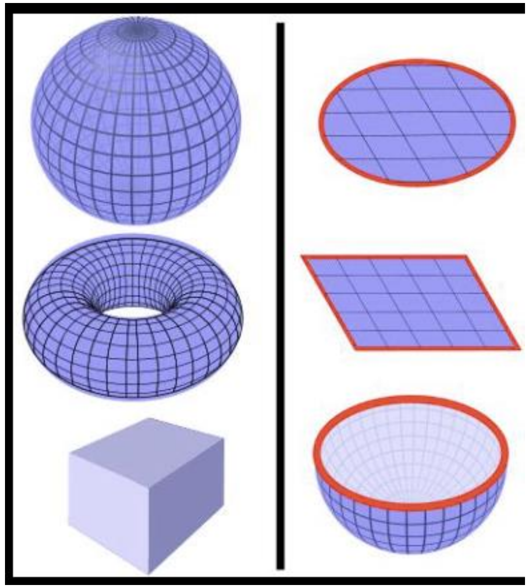
Manifolds

- What Is a Manifold?
- Locally Euclidean
- Local versus Global
- Doughnut and Coffee Cup



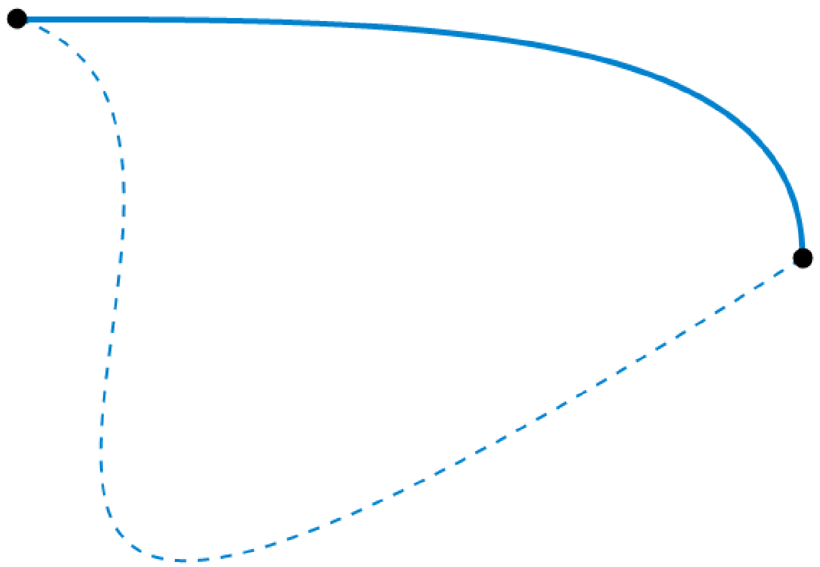


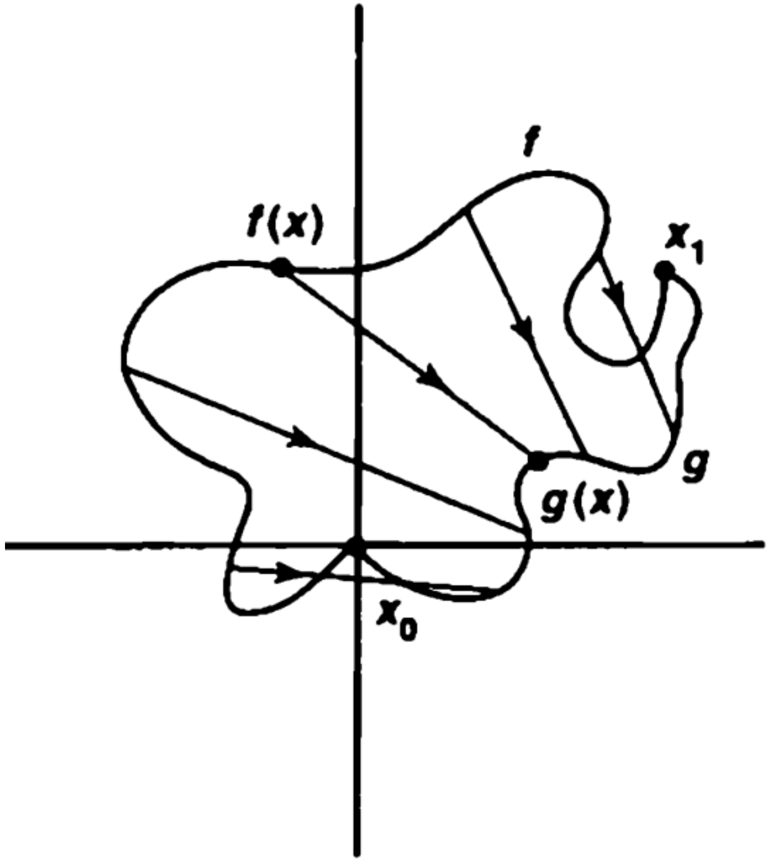




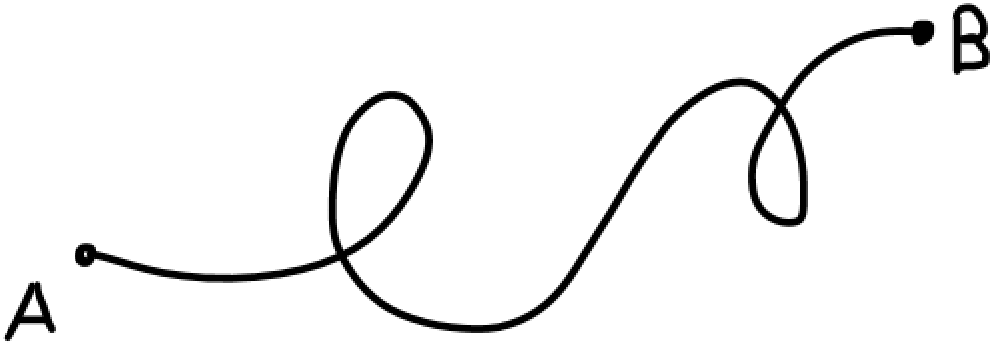
Homotopy Theory

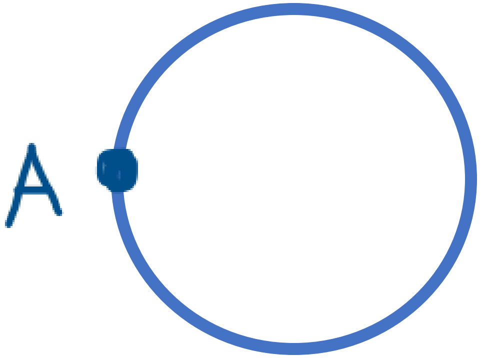
- Two continuous functions from one topological space to another are called homotopic if one can be continuously deformed into the other. This deformation is called a homotopy.

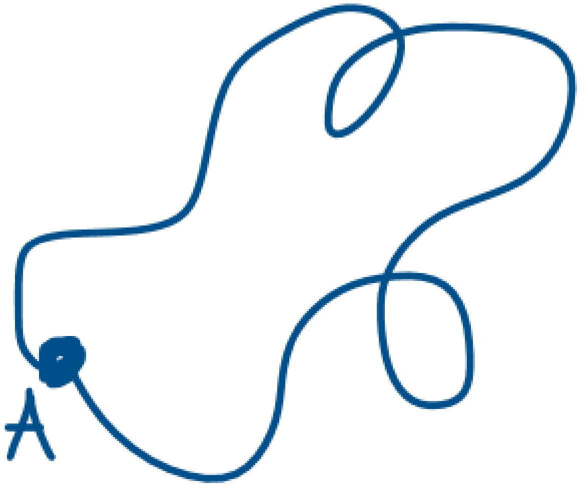


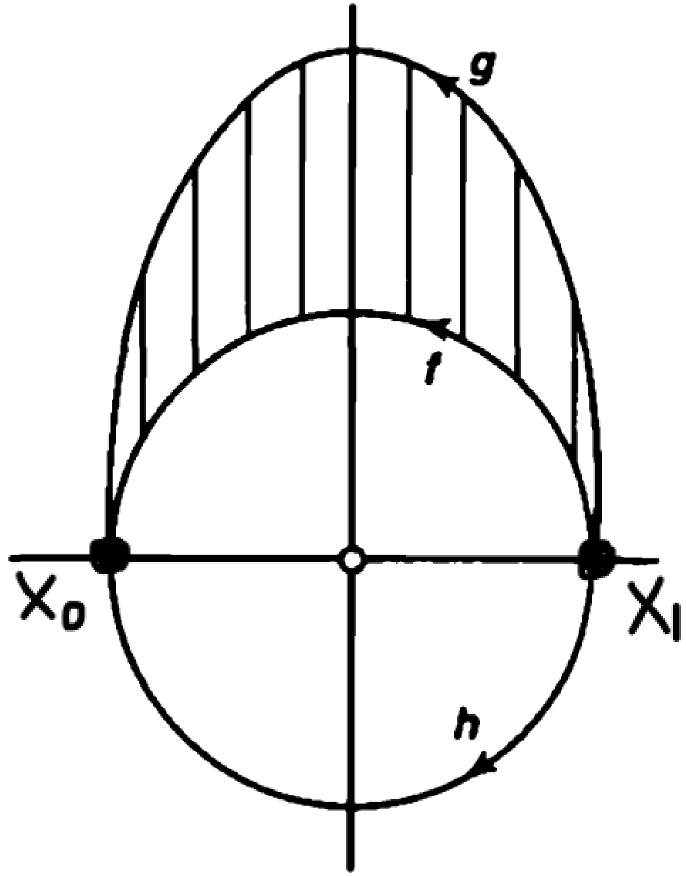


$$F(x,\ t) = (1 - t)f(x) + tg(x)$$

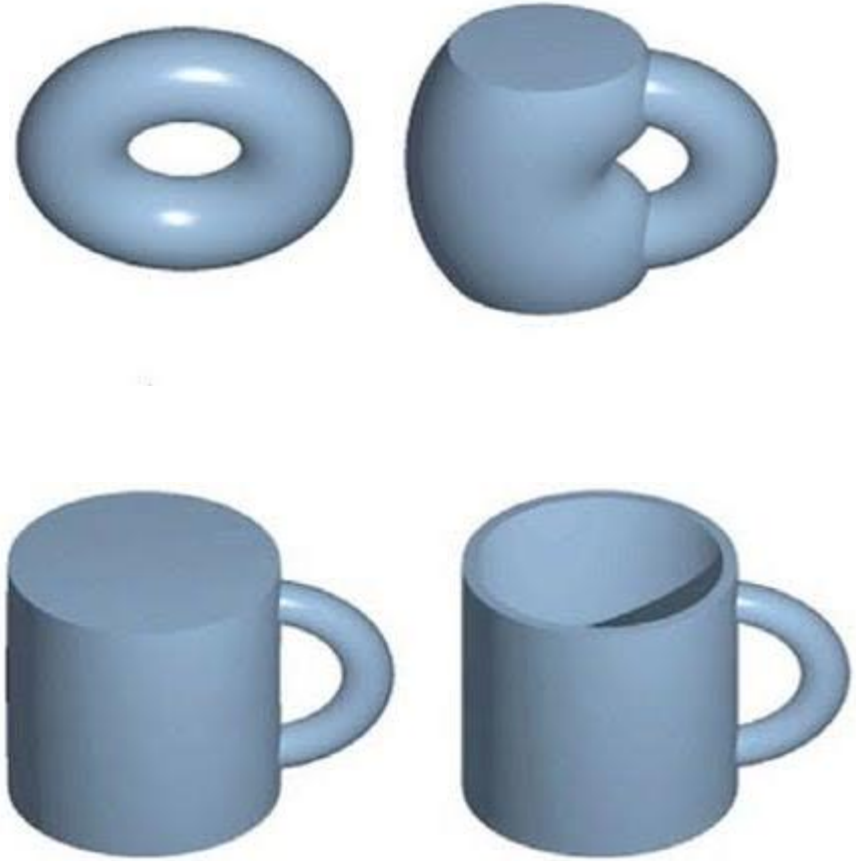








$\mathbb{R}^2 \setminus \{0\}$



Groups

A group is a set G with a binary operation that satisfies:

- Associativity: For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Identity: There exists $e \in G$ such that for all $g \in G$, $g \cdot e = e \cdot g = g$
- Inverse: For every $g \in G$, there exists $h \in G$ such that $g \cdot h = h \cdot g = e$.

Group Presentation

A **group presentation** describes a group by:

- Generators (S): A set of elements that generate the group
- Relations (R): Equations that the generators satisfy

Notation: $\langle S \mid R \rangle$

Example: Cyclic Group C_n $\langle a \mid a^n = 1 \rangle$

Direct Product of groups

Let $(A,\cdot)$ and $(B,\star)$ be groups.

We define the *direct product* of A and B to be the set

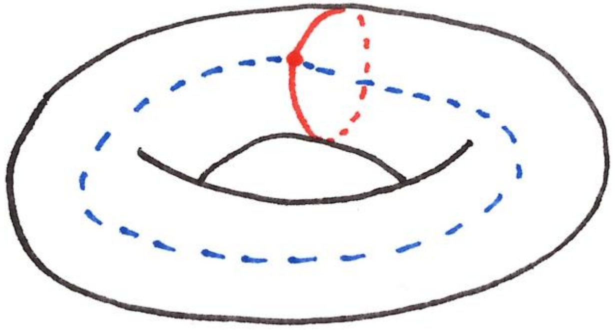
$$A \times B = \{(a,b) : a \in A, b \in B\}$$

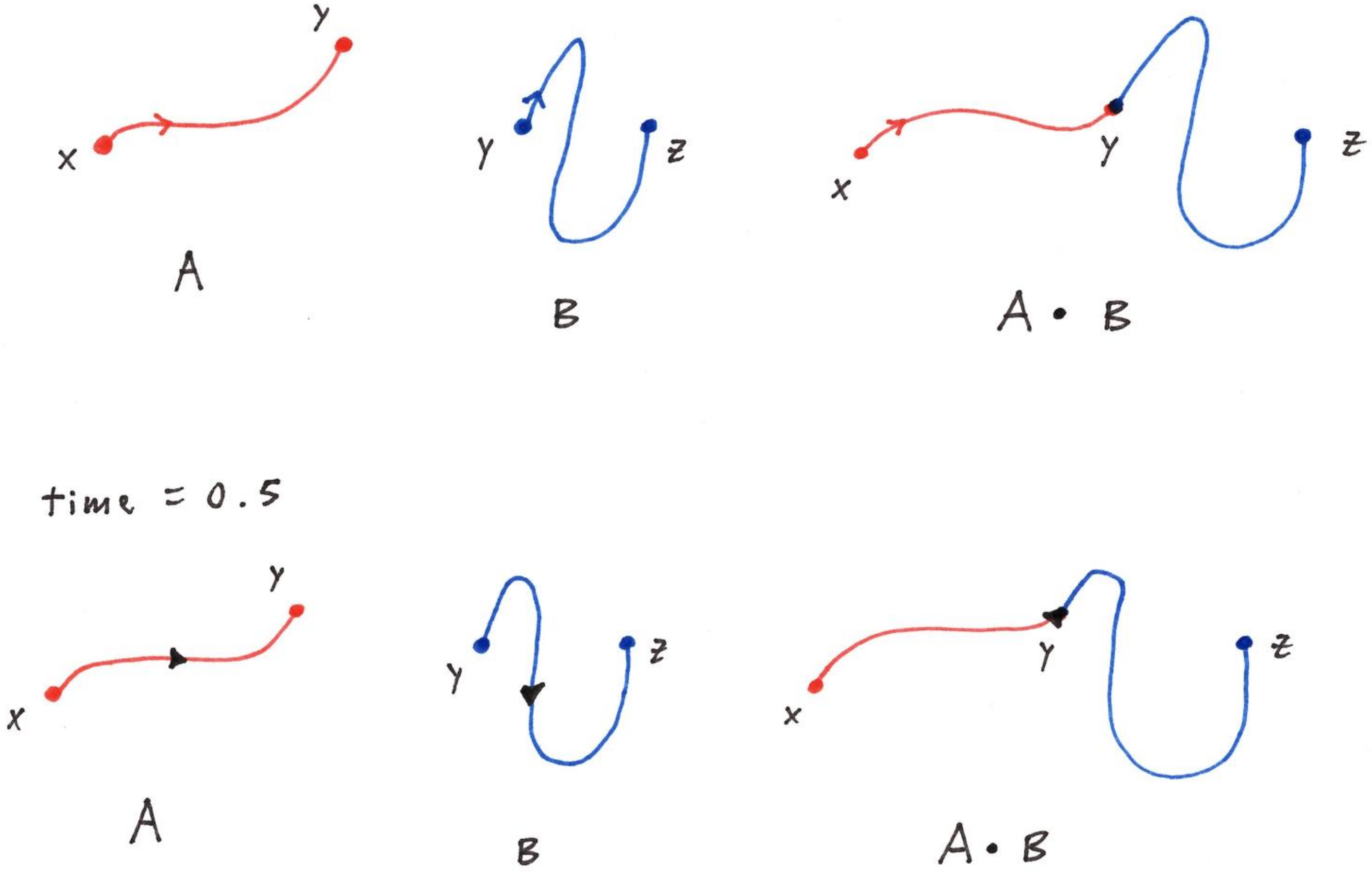
equipped with the operation

$$(a1,b1)(a2,b2) = (a1 \cdot a2, b1 \star b2).$$

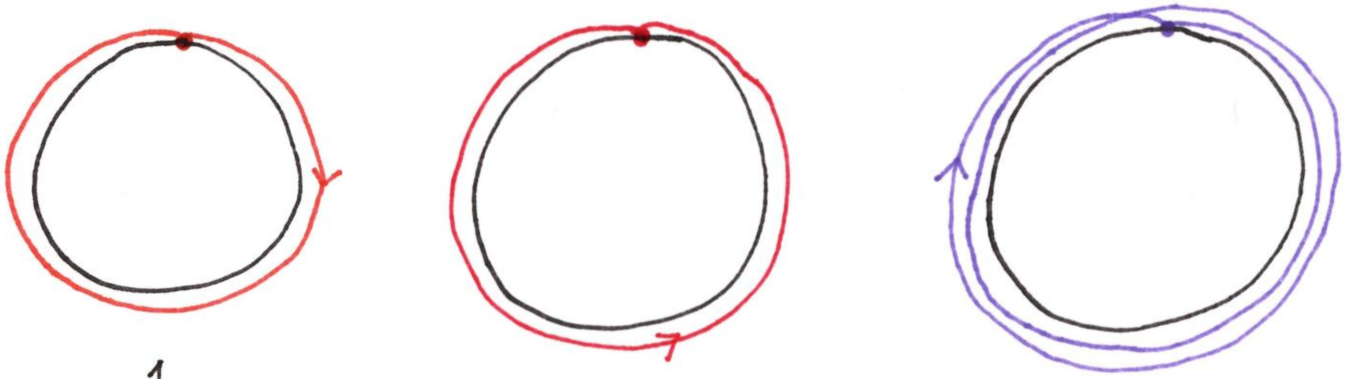
Fundamental Groups

- Elements: Path classes of loops based at a given point
- Operation: Path multiplication



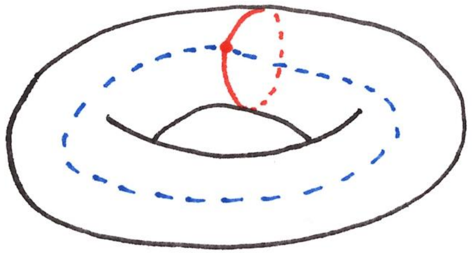


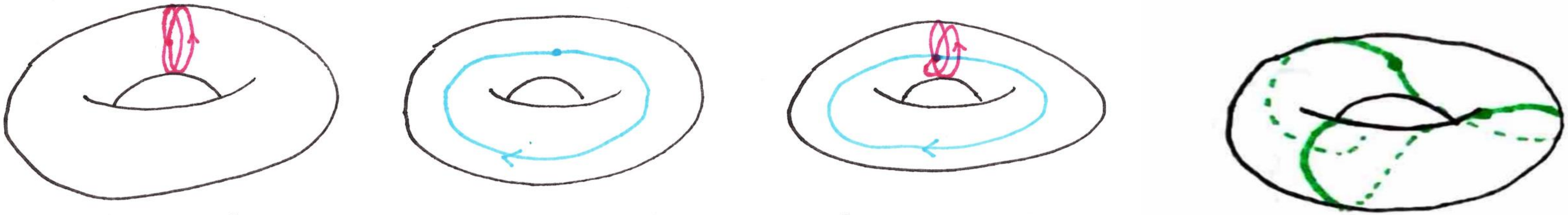
Circle: $\mathbb{Z}$



1   −1   2

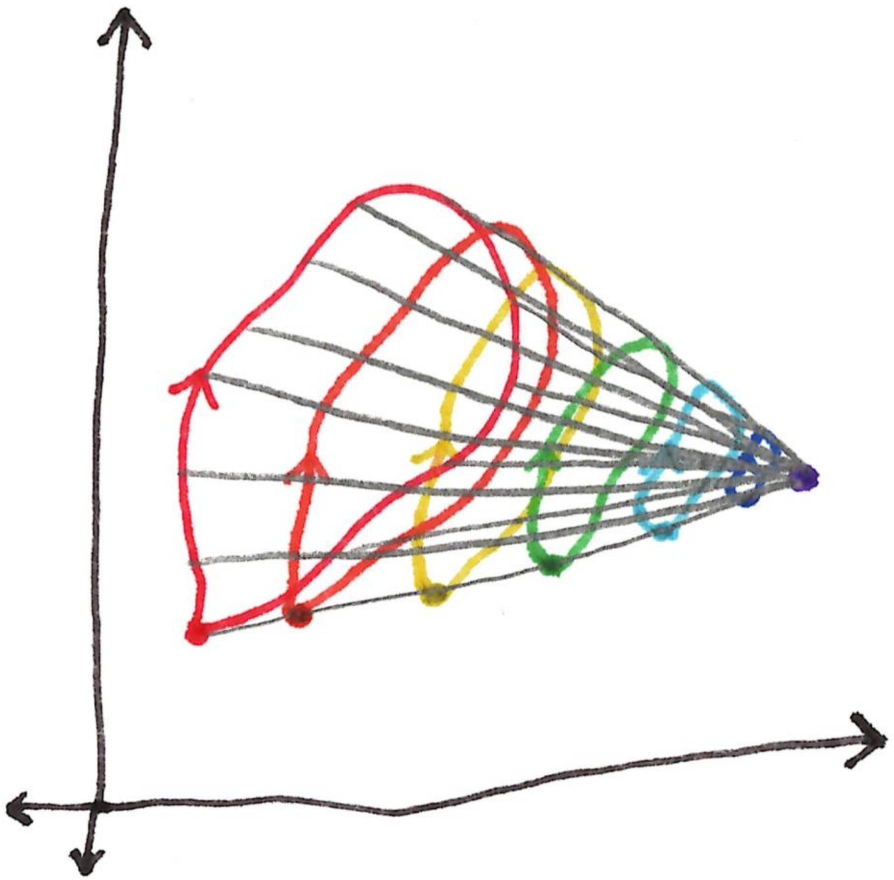
Torus: $\mathbb{Z} \times \mathbb{Z}$



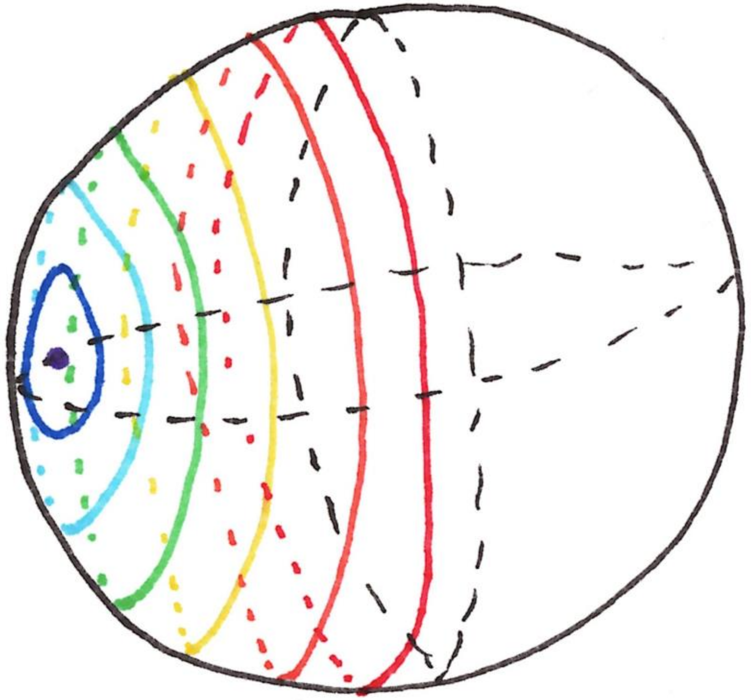


(−2, 0)   (0, 1)   (−2, 1)   Also (−2, 1)

Plane: Trivial group

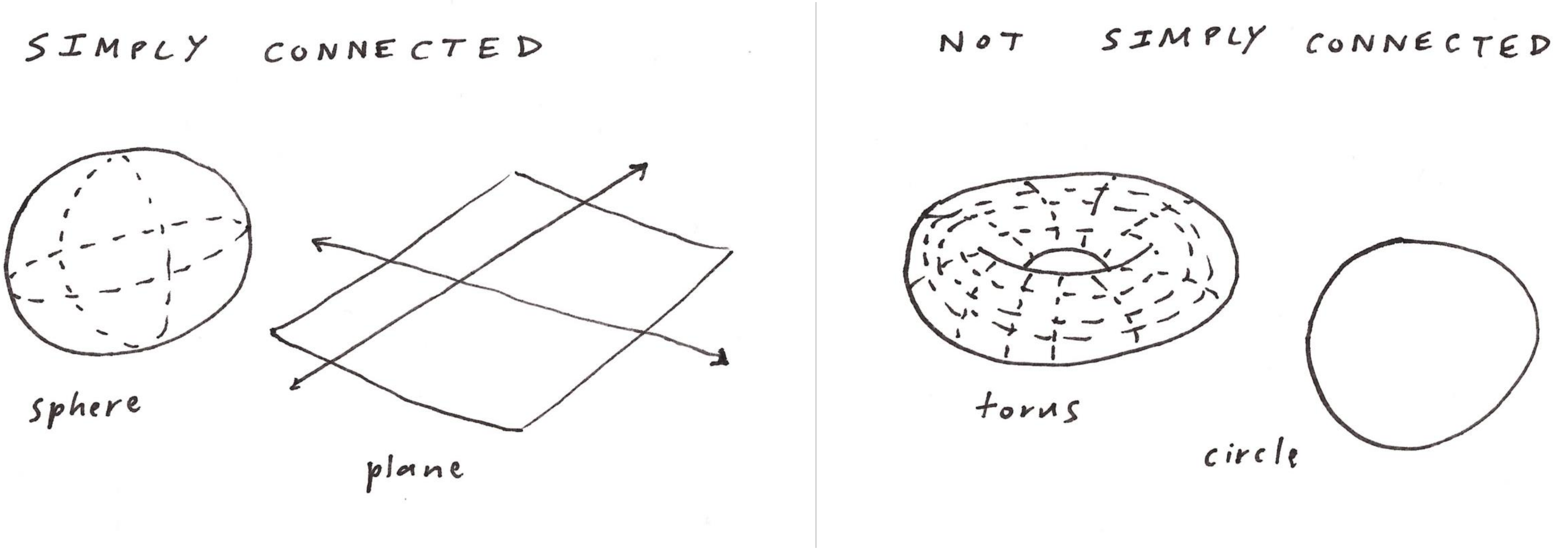


Sphere: Trivial group



Simply Connected

- Fundamental group is trivial



Conclusion

- Every 3-dimensional manifold which is closed, connected, and simply connected is homeomorphic to the three-dimensional sphere