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Introduction to Semigroup Theory for Differential Equations

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Outline

1. Motivation
2. Preliminaries
   - Functional Analysis
3. Semigroup Theory
   - C_0-Semigroup
   - Infinitesimal Generators of the Semigroup
   - Differential Properties of the Semigroup
   - Properties of Generators
   - The Hille-Yosida Theorem
4. Conclusion
5. References

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Consider the 1D ODE given by

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- Interestingly, the family $\{e^{At}\}_{t\geq 0}$ of bounded linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, for all $t \geq 0$, enjoys the algebraic properties of a Semigroup under composition; that is, $e^0 = I$ and $e^{A(t+s)} = e^{At} \cdot e^{As}$.

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- When studying differential equations on infinite-dimensional abstract function spaces, one may therefore expect analogous properties to hold for *well-posedness*.

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- In these partial differential equations, A is a differential operator rather than a matrix, and it may be unbounded.
- Semigroup theory provides a natural framework for analyzing abstract differential equations on infinite-dimensional spaces.
- In particular, we study the *existence* and *uniqueness* of a solution $\mathbf{u} : [0, \infty) \to X$, where X can be an abstract space (e.g., a Banach or Hilbert space), of the first-order ODE (or a PDE formulated as an ODE)

$$\begin{cases} \dot{\mathbf{u}}(t) = A\,\mathbf{u}(t), & (t \geq 0), \\ \mathbf{u}(0) = u, \end{cases} \tag{3}$$

where $u \in X$ is given, and A is a linear operator with domain $D(A)$.

Normed Linear Spaces

Definition (Normed Linear Space)

A *norm* on a vector space X is a map $\|\cdot\| : X \to [0, \infty)$ such that:

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||\alpha x|| = |\alpha| \; ||x||, \quad \alpha \in \mathbb{K}$
4. $||x + y|| \leq ||x|| + ||y||$

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- continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that:

$$\|Tx - Tx_0\|_Y < \varepsilon \quad \text{for all } x \text{ satisfying } \|x - x_0\| < \delta.$$

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For a linear operator, bounded $\iff$ continuous.

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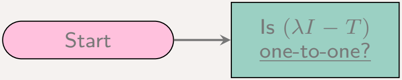

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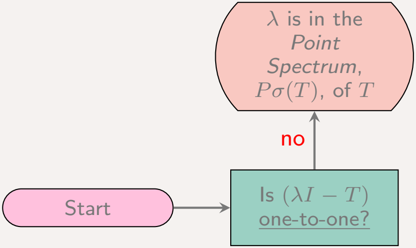

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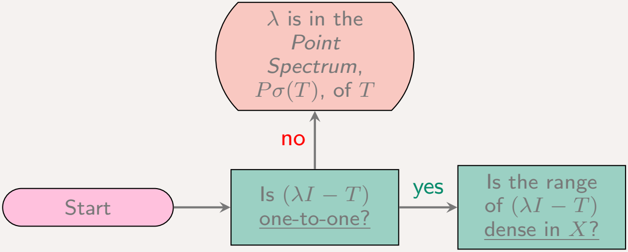

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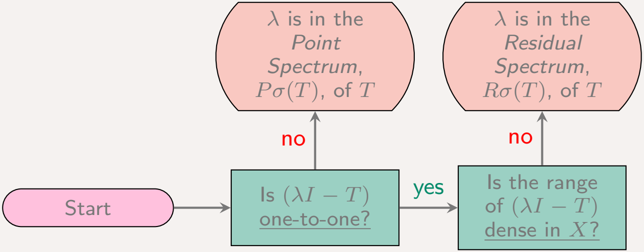

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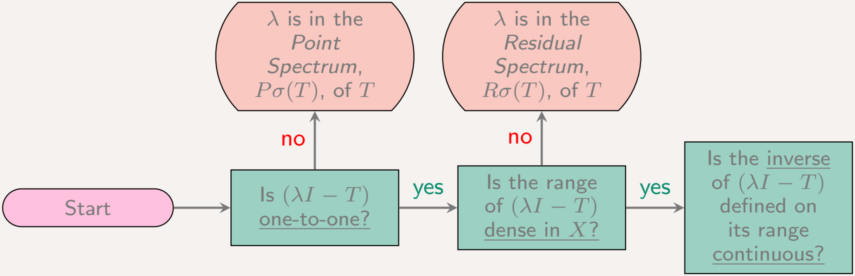

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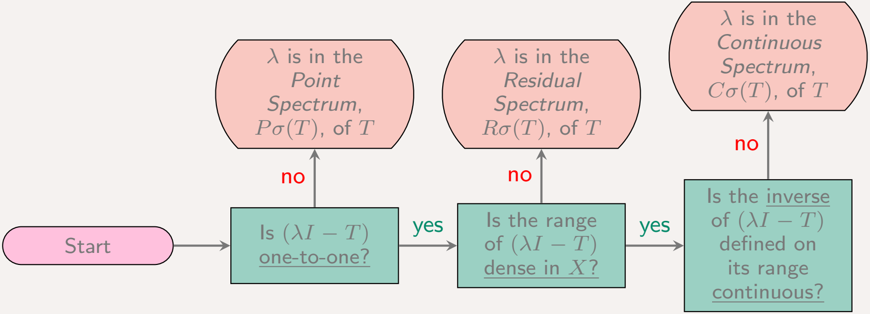

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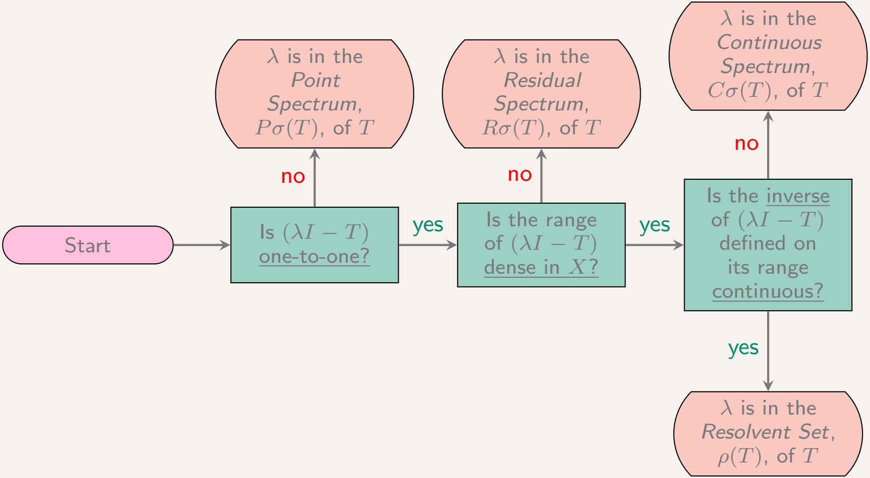

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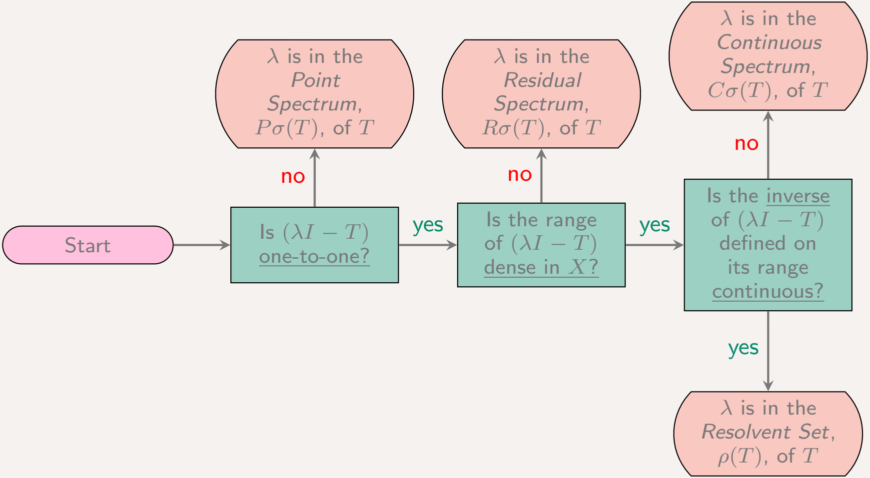

If $\lambda \in \rho(T)$, then resolvent operator of A is defined as:

$$R(\lambda, T) := (\lambda I - T)^{-1}$$

Semigroup

Definition (Strongly Continuous or C_0-Semigroup)

A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators mapping from X to X which satisfies:

- $S(0) = I$
- $S(s+t) = S(s)S(t), \quad (s, t \leq 0)$
- For each $u \in X$ the mapping $t \mapsto S(t)u$ is continuous from $[0, \infty) \to X$

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Definition (Uniformly Continuous Semigroups)

A Semigroup $\{S(t)\}_{t\geq 0}$ is uniformly continuous if: $\lim_{t\to 0} ||S(t) - I|| = 0$.

Infinitesimal Generators of the Semigroup

Definition (Infinitesimal Generator, (Definition 7.4.1[a]))

[a] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2nd ed., American Mathematical Society, Providence, RI, 2010.

Given a strongly continuous Semigroup $\{S(t)\}_{t\geq 0}$, we call $A : D(A) \to X$ the *infinitesimal generator* of the Semigroup, where $D(A)$ is the domain of A such that:

$$D(A) := \left\{ u \in X : \lim_{t\to 0+} \frac{S(t)u - u}{t} \text{ exists} \right\}$$

then we have:

$$Au := \lim_{t\to 0+} \frac{S(t)u - u}{t} \quad (u \in D(A))$$

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For example, in n-D ODEs (2), matrix A is generator of the Semigroup $\{e^{At}\}_{t\geq 0}$.

Differential Properties of the Semigroup

Theorem (Theorem 7.4.1[a])

[a] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2nd ed., American Mathematical Society, Providence, RI, 2010.

Let $u \in D(A)$, then we have the following properties:

1. $S(t)u \in D(A)$ *for each* $t \geq 0$
2. $AS(t)u = S(t)Au$ *for each* $t \geq 0$
3. *The mapping* $t \mapsto S(t)u$ *is differentiable for each* $t > 0$
4. $\frac{d}{dt}S(t)u = A\,S(t)u, \quad (t > 0)$

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1. 1-2 can be proved directly by invoking the definition of $D(A)$ and using the semigroup property:

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2. 3-4 can be proven using the limit definition of the derivative:

$$\lim_{h\to 0+} \frac{S(t+h)u - S(t)u}{h} = S(t) \lim_{h\to 0+} \frac{S(h)u - u}{h} = S(t)Au = AS(t)u.$$

Properties of Generators

Theorem (Theorem 7.4.2[a])

[a] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2nd ed., American Mathematical Society, Providence, RI, 2010.

Let $A : D(A) \to X$ generate a contraction Semigroup. Then

1. *The domain $D(A)$ is dense in X.*
2. *A is a closed operator.*

Note: *A is a closed operator if, for every sequence $u_k \in D(A)$ such that $u_k \to u$ and $Au_k \to v$, we have $u \in D(A)$ and $v = Au$.*

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$$u^t := \int_0^t S(s)u\,ds.$$

Then note that $u^t/t \to u$ as $t \to 0^+$, and moreover $u^t \in D(A)$.

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2. Let $u_k \in D(A), (k = 1, ...), u_k \to u, Au_k \to v$, and use:

$$S(t)u_k - u_k = \int_0^t S(s)Au_k\,ds \overset{k\to\infty}{\Longrightarrow} S(t)u - u = \int_0^t S(s)v\,ds.$$

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Divide by t and let $t \to 0^+$ to obtain $Au = v$.

The Hille-Yosida Theorem

Theorem (Theorem 7.4.4[a]; Theorem 3.1[b])

[a] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2nd ed., American Mathematical Society, Providence, RI, 2010.

[b] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, 1983.

A linear (unbounded) operator A is the infinitesimal generator of a C_0 Semigroup of contractions $S(t), t \geq 0$, if and only if:

1. *A is closed and $\overline{D(A)} = X$.*
2. *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$ we have:*

$$||R(\lambda, A)|| \leq \frac{1}{\lambda}. \tag{4}$$

The Hille-Yosida Theorem

Proof idea:

$$A_\lambda = -\lambda I + \lambda^2 R(\lambda, A)$$

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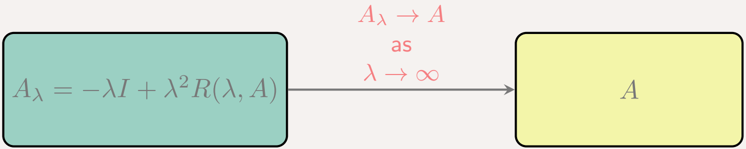

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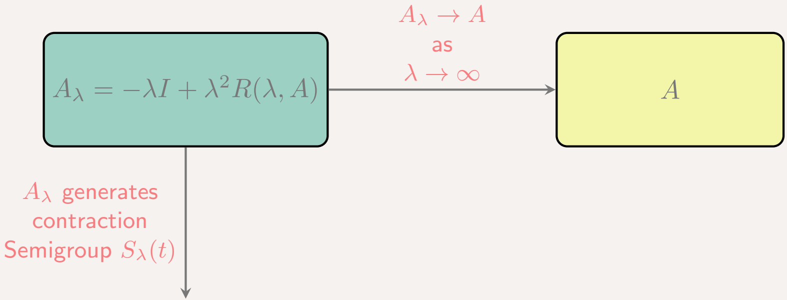

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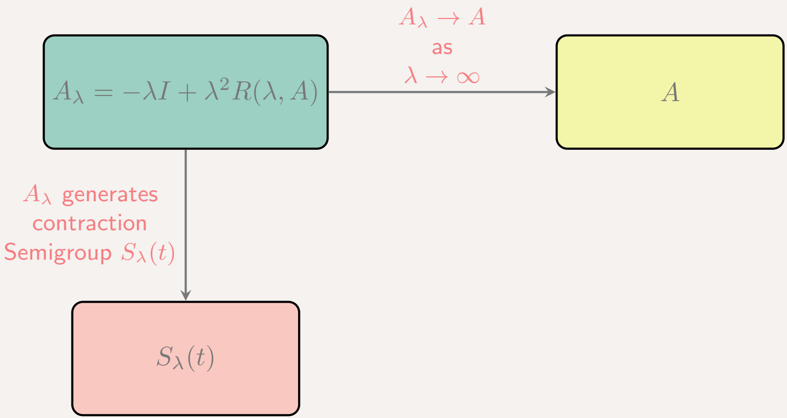

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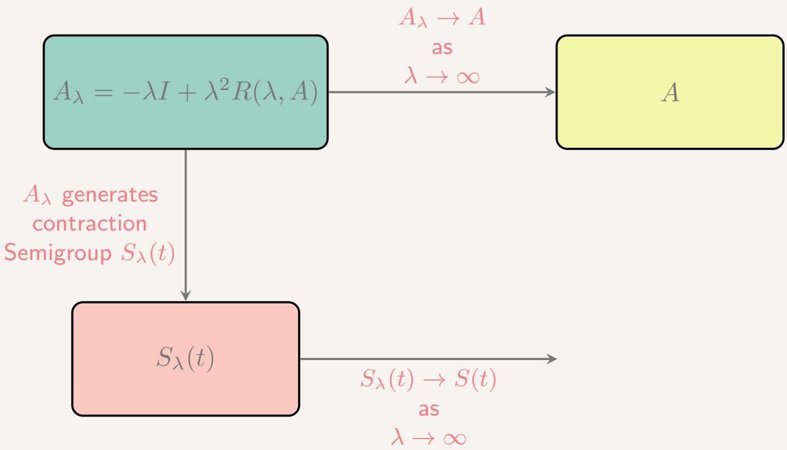

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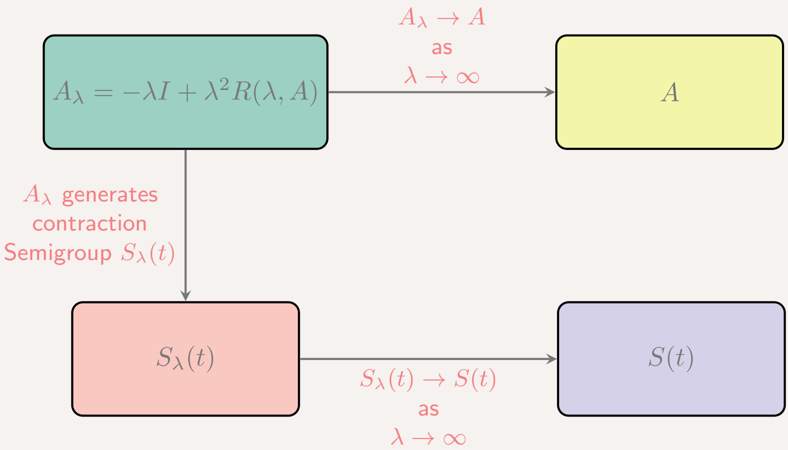

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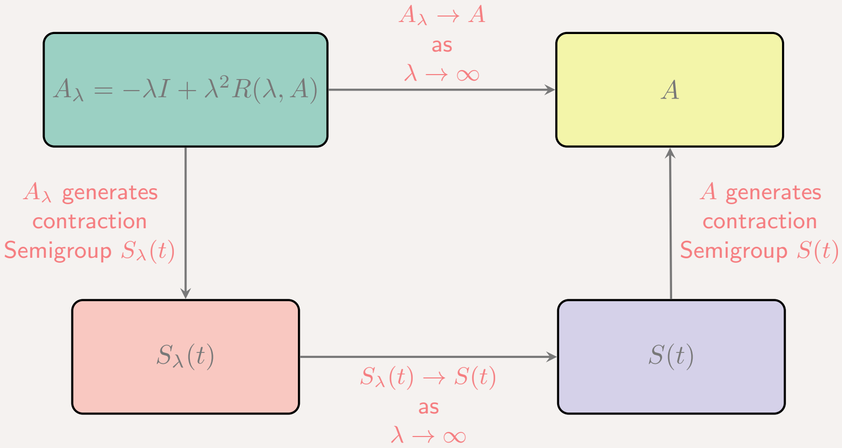

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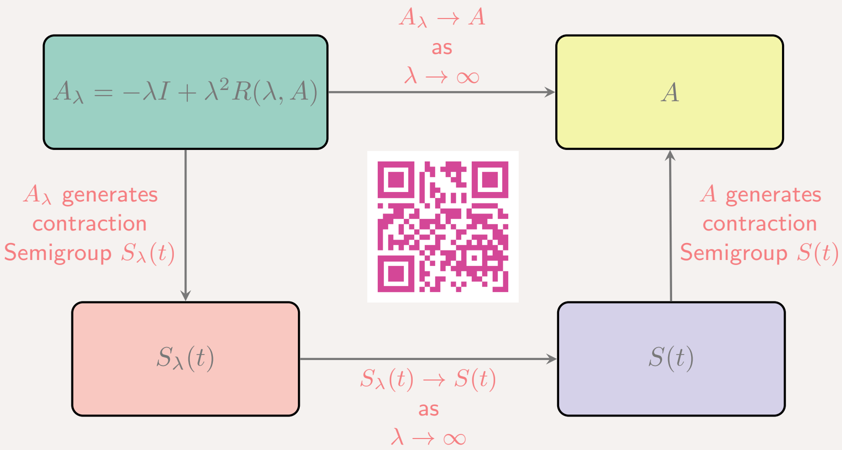

Conclusion and Future Work

From this project, we have learned that Semigroup theory serves as a powerful tool for studying the well-posedness of ODEs/PDEs.

- This talk covered only linear C_0-Semigroups and the Hille-Yosida generation theorem.
- Future directions:
  1. Regularity properties of Semigroups
  2. Stability properties of Semigroups
  3. Applications to: Parabolic Equations (L^2, L^p Theory), Wave/heat equation, Schrödinger equations, Korteweg-de Vries Equation, etc.

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Thank you for your kind attention!