Amenable and Property (T) Discrete Groups

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- We define the operator spaces
$\ell^1(G) \coloneqq \left\{ f : G \to \mathbb{C} \;\middle|\; \|f\|_1 = \sum_{g \in G} |f(g)| < \infty \right\},$
$\ell^2(G) \coloneqq \left\{ f : G \to \mathbb{C} \;\middle|\; \|f\|_2^2 = \sum_{g \in G} |f(g)|^2 < \infty \right\},$
$\ell^\infty(G) \coloneqq \left\{ f : G \to \mathbb{C} \;\middle|\; \|f\|_\infty = \sup_{g \in G} |f(g)| < \infty \right\}.$[1]

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Operator spaces on ℓ^2

$\mathcal{B}(\ell^2(G))$ is the space of bounded operators on $\ell^2(G)$.
$\mathcal{U}(\ell^2(G))$ is the space of unitary ($U^* = U^{-1}$) operators on $\ell^2(G)$.

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Unitary Representations

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Unitary representation of G

A unitary representation $\pi : G \mapsto \mathcal{U}(\ell^2(G))$ is a map that sends $g \to \pi_g(x)$ such that

$$\langle \pi_g \delta_h, \pi_g \delta_h \rangle = \langle \delta_h, \delta_h \rangle, \quad \forall g, h \in G.$$

Representations & Actions

- Special case of a unitary representation: **left regular rep.** is a map $\lambda : G \mapsto \mathcal{U}(\ell^2(G))$ where

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$$(g \cdot \zeta)(x) \; = \; \zeta(g^{-1}x) \quad \text{for } x \in G$$

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$$(g \cdot \zeta)(x) \;=\; \zeta(g^{-1}x) \quad \text{for } x \in G$$

- We say $f : \ell^\infty(G) \to \mathbb{C}$ is *G*-**invariant** if $f(g \cdot \zeta) = f(\zeta)$.

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Amenable Groups

- A **state** $\varphi : \ell^\infty(G) \to \mathbb{C}$ is a (continuous) linear functional such that $\varphi(\zeta) \geq 0$ whenever $\zeta \geq 0$, $\varphi(\zeta^*) = \overline{\varphi(\zeta)}$, and $\varphi(1) = 1$.

Definition

Let G be a discrete group. We say G is **amenable** when there exists a state $\varphi : \ell^\infty(G) \to \mathbb{C}$ such that

$$\varphi(g \cdot \zeta) = \varphi(\zeta) \quad \forall g \in G,\ \forall \zeta \in \ell^\infty(G),$$

that is, φ is a G-invariant state on $\ell^\infty(G)$.

- An equivalent definition is that there exists a kind of *mean* on G - hence G is a<u>mean</u>able!
  - $\hookrightarrow$ Speaking of equivalent definitions...

Equivalent Definitions

There are many equivalent definitions for amenability.

- G has **approximate invariant means** when there exists a sequence $(\mu_n)_n$ of probability measures[1] such that

$$\forall g \in G,\ \|\mu_n - g \cdot \mu_n\|_1 \to 0.$$

- A **Følner sequence** is a sequence $(F_n)_n$ of finite subsets of G such that $\forall g \in G$,

$$\frac{|g \cdot F_n \ \triangle \ F_n|}{|F_n|} \to 0.$$

- A representation λ has **almost invariant vectors**, notated $1_G \prec \lambda$, iff there exists a sequence of $(\zeta_n)_n$ over $\ell^2(G)$ such that

$$\|\lambda_g(\zeta_n) - \zeta_n\| \to 0, \quad \forall g \in G.$$

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[1] A function that assigns a probability to each group element

Equivalent Definitions of Amenability

Let G be a discrete group. The following are equivalent:

1. G is amenable;
2. There exists a G-invariant state φ on $\ell^\infty(G)$;
3. G has approximate invariant means, i.e. $\|\mu_n - g \cdot \mu_n\|_1 \to 0$;
4. There exists a Følner sequence $(F_n)_n$ over G;
5. The left regular representation λ has approximate invariant vectors, i.e. $1_G \prec \lambda$.

- Proving equivalence between all these definitions takes a while, so let's consider an example and prove it a few ways.

An Example: $(\mathbb{Z}, +)$

Prop: $(\mathbb{Z}, +)$ is amenable and has a Følner sequence.
Sketch of Proof. Take $F_n := \{-n, -(n-1), ..., n-1, n\}$, then for $m \in \mathbb{Z}$ we have that $(m + F_n) = \{m - n, ..., m + n - 1, m + n\}$. Thus

$$(m + F_n) \triangle F_n = \{-n \leq k \leq m + n : k > n \text{ or } k < m - n\}.$$

Now, we can assume WLOG that $n > m$ since we will take $n \to \infty$. For $m > 0$, we have $|(m + F_n) \triangle F_n| = 2m$. And, for $m < 0$ consider the automorphism $\phi : \mathbb{Z} \mapsto \mathbb{Z}$, $\phi(x) = -x$, and note that ϕ preserves the symmetric difference. Thus, for $m < 0$, we can write
$(m + F_n) = \phi(-m)F_n$, and

$$|(m + F_n) \triangle F_n| = |\phi((m + F_n) \triangle F_n)| = |\phi(m)F_n \triangle F_n|,$$

so, for any $m \in \mathbb{Z}$,

$$|(m + F_n) \triangle F_n| = 2m.$$

An Example: $(\mathbb{Z}, +)$

And, since $|F_n| = 2n + 1$, as $n \to \infty$, we have

$$\frac{|(m + F_n) \triangle F_n|}{|F_n|} = \frac{2|m|}{2n + 1} \to 0. \quad \square$$

We now show amenability by constructing an invariant state on $\ell^\infty(\mathbb{Z})$. Define $\mu_n : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ by

$$\mu_n(f) := \frac{1}{|F_n|} \sum_{x \in F_n} f(x) = \frac{1}{2n + 1} \sum_{x=-n}^{n} f(x),$$

and notice that μ_n is a state since

$$\mu_n(1) = \frac{1}{2n + 1} \sum_{x=-n}^{n} 1 = \frac{2n + 1}{2n + 1} = 1$$

and whenever $f \geq 0$, we have that $\sum_{k=-n}^{n} f(k) \geq 0$, hence $\mu_n(f) \geq 0$. For any $n \in \mathbb{Z}$:

An Example: $(\mathbb{Z}, +)$

$$\mu_n(m+f) - \mu_n(f) = \frac{1}{2n+1}\left(\sum_{k=-n}^{n} f(m+k) - \sum_{k=-n}^{n} f(k)\right)$$

$$= \frac{1}{2n+1}\left(\sum_{k=m-n}^{m+n} f(k) - \sum_{k=-n}^{n} f(k)\right) = \frac{1}{2n+1}\left(\sum_{k\in(m+F_n)\triangle F_n} f(k)\right)$$

$$\leq \frac{1}{2n+1}\left(|(m+F_n)\triangle F_n|\right) = \frac{2|m|\,\|f\|_\infty}{2n+1}.$$

Therefore, as $n \to \infty$, we have that $\mu_n(m+f) - \mu_n(f) \to 0$. Note that, since each μ_n is a state, $\|\mu_n\|_{\text{op}} = 1$. Since μ_n has a cluster point μ, we have $\mu(m+f) = \mu(f)$, so μ is an invariant state on $\ell^\infty(\mathbb{Z})$. $\square$

- Classical definition: amenable groups do not admit a ”paradoxical decomposition”

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Proof. Write

$$\begin{aligned}
A^+ &= \{\text{reduced words beginning with } a\}, \\
A^- &= \{\text{reduced words beginning with } a^{-1}\}, \\
B^+ &= \{\text{reduced words beginning with } b\}, \\
B^- &= \{\text{reduced words beginning with } b^{-1}, \} \\
C &= \{\varepsilon, b, b^2, \cdots\}.
\end{aligned}$$

F_2 *is not amenable.*

Let $\sqcup$ denote the disjoint union. Decompose F_2 as

$$F_2 = \begin{cases} A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C) \\ A^+ \sqcup aA^- \\ b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C). \end{cases}$$

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Suppose towards a contradiction that μ is an amenable state. Let $\mathbb{1}_A$ be the characteristic indicator on A, then

$$\begin{aligned} \mu(1) &= \mu(\mathbb{1}_{A^+} + \mathbb{1}_{A^-} + \mathbb{1}_{(B^+\setminus C)} + \mathbb{1}_{(B^-\cup C)}) \\ &= \mu(\mathbb{1}_{A^+}) + \mu(a \cdot \mathbb{1}_{A^-}) + \mu(b^{-1} \cdot \mathbb{1}_{(B^+\setminus C)}) + \mu(\mathbb{1}_{(B^-\cup C)}) \\ &= \mu(\mathbb{1}_{A^+}) + \mu(\mathbb{1}_{a\cdot A^-}) + \mu(\mathbb{1}_{b^{-1}\cdot(B^+\setminus C)}) + \mu(\mathbb{1}_{(B^-\cup C)}) \\ &= \mu(\mathbb{1}_{A^+} + \mathbb{1}_{a\cdot A^-}) + \mu(\mathbb{1}_{b^{-1}\cdot(B^+\setminus C)} + \mathbb{1}_{(B^-\cup C)}) \\ &= \mu(1) + \mu(1) = 2\mu(1), \quad \text{so} \quad 1 = 2. \end{aligned}$$

Thus F_2 cannot be amenable. ■

Examples of Amenable Groups

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5. If, for every finitely generated $H \leq G$, we have H amenable, then G is amenable.

- Note the contrapositive to (3): If H is not amenable and $H \leq G$, then G is not amenable.

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We say π has nonzero invariant vectors when there exists a vector $\zeta \neq 0$ such that

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Let G be a group. We say G **has Property (T)** if, whenever a representation π has almost invariant vectors, then it has nonzero invariant vectors.

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Definition

Let G be a group. We say G **has Property (T)** if, whenever a representation π has almost invariant vectors, then it has nonzero invariant vectors.We also write $1_G \prec \pi \implies 1_G \subseteq \pi$, and may say *$G$ has (T).*

Theorem: If G is a countable (T) group, then G is finitely generated

Sketch of Proof. Let $\mathcal{C}$ be the set of all finitely generated subgroups of G, and since G is countable, we have

$$G = \bigcup_{H \in \mathcal{C}} H.$$

For $H \in \mathcal{C}$, construct the representation

$$\pi := \bigoplus_{H \in \mathcal{C}} \lambda_{G/H},$$

where $\lambda_{G/H} : G \mapsto \mathcal{U}(\ell^2(G/H))$ acts by translation. Note that this is a unitary rep, $\pi : G \mapsto \mathcal{U}(\mathcal{J})$ where

$$\mathcal{J} = \bigoplus_{H \in \mathcal{C}} \ell^2(G/H),$$

Theorem Continuation

i.e. $\mathcal{J}$ is the smallest Hilbert space containing mutually orthogonal copies of $\ell^2(G/H)$ for each finitely generated subgroup. Now, we will show that π has almost invariant vectors. Let $F \subset G$, finite, and let $K \in \mathcal{C}$ be the subgroup generated by F, then $\forall g \in F$,

$$\|\pi(g)\delta_K - \delta_K\| = 0,$$

where δ_K is viewed as a vector in $\mathcal{J}$. Since G is a (T) group, $\exists \zeta \in \mathcal{J}$ a nonzero invariant vector, and

$$\zeta = \bigoplus_{H \in \mathcal{C}} \zeta_H,$$

where ζ_H is the orthogonal projection of ζ onto the subspace $\ell^2(G/H)$.

Theorem Continuation

Suppose $H \in \mathcal{C}$ is such that $\zeta_H \neq 0$, then ζ_H is a nonzero $G-$invariant vector in $\ell^2(G/H)$, so ζ_H must be a constant sequence indexed by G/H of nonzero complex numbers. If G/H is not finite, this implies it does not lie in $\ell^2(G/H)$, thus G/H must be finite.[2] Therefore, since H is finitely generated, G is finitely generated. $\square$

[2] More on this later...

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Theorem

Let G have Property (T) and be amenable. Then G is finite.

Proof: Since G is amenable, λ has an almost invariant vector. By Property (T), we are guaranteed that λ has a nonzero G-invariant vector, say $\zeta \in \ell^2(G)$. This means that

$$\zeta_h = \lambda(g)\zeta_h = \zeta_{g^{-1}h}$$

for all $g, h \in G$. It follows from this that $\zeta_h = \zeta_g$ for all $g, h \in G$ and therefore ζ is a constant sequence. Since ζ is nonzero then for any $g \in G$ we take $\alpha = |\zeta_g|^2 > 0$, if we suppose that G is an infinite group then we would have that

$$\|\zeta\|_2^2 = \sum_{g\in G} |\zeta_g|^2 = \sum_{g\in G} \alpha = \infty$$

and therefore $\zeta \notin \ell^2(G)$, a contradiction. Thus G is finite. ■

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Questions