Nonstandard Analysis

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Properties

- $(\mathbb{R}^*, +, \cdot, 0, 1, <)$ is an ordered field which contains $(\mathbb{R}, +, \cdot, 0, 1, <)$
- There exists $\varepsilon \in \mathbb{R}^*$ such that $0 < \varepsilon < r$ for all $r > 0$
- For all $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}$, there is a "natural" extension, $f : (\mathbb{R}^*)^n \to \mathbb{R}^*$. In particular, the extensions of $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ coincide with the field operations on $\mathbb{R}^*$
- Any statement expressible in first-order logic and mentioning only standard numbers is true in $\mathbb{R}$ if and only if it is true in $\mathbb{R}^*$.

Notation

- $\mathbb{R}_{\text{fin}} = \{x \in \mathbb{R}^* : |x| < n \text{ for some } n \in \mathbb{N}\}$
- $\mathbb{R}_{\text{inf}} = \{x \in \mathbb{R}^* : |x| > n \text{ for all } n \in \mathbb{N}\} = \mathbb{R}^* \setminus \mathbb{R}_{\text{fin}}$
- $\mu = \{x \in \mathbb{R}^* : |x| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$

Note that μ is an ideal of $\mathbb{R}_{\text{fin}}$, and $\mathbb{R}_{\text{fin}}$ is a subring of $\mathbb{R}^*$

Definition

Definition: Infinitely Close

$r, s \in \mathbb{R}^*$ are infinitely close, denoted $r \approx s$, if $r - s \in \mu$

Note that $\approx$ is an equivalence relation, that is for all $r, s, t \in \mathbb{R}^*$,

- $r \approx r$
- if $r \approx s$, then $s \approx r$
- if $r \approx s$ and $s \approx t$, then $r \approx t$

Theorem

Theorem: Existence of Standard Parts

For all $r \in \mathbb{R}_{\text{fin}}$, there exist a unique $s \in \mathbb{R}$ such that $r \approx s$

Proof: First of all, if $r \approx s_1$ and $r \approx s_2$ with $s_1, s_2 \in \mathbb{R}$, then $s_1 - s_2 \in \mathbb{R} \cap \mu$, and thus, $s_1 - s_2 = 0$, so uniqueness is proved. To show existence, let $S = \{x \in \mathbb{R} : x \leq r\}$. Since $r \in \mathbb{R}_{\text{fin}}$, S is bounded. Let $s = \sup(S)$

Proof Continued

For any $\delta \in \mathbb{R}^{>0}$, $s + \delta \notin S$ since $s + \delta > s > x$ for all $x \in S$. Thus, $s + \delta > r$. If we also had $s - \delta \geq r$, $s - \delta$ would be an upper bound for S, which contradicts s being the least upper bound of S, so we must have $s - \delta < r$. Thus, $|r - s| < \delta$, and since this holds for any $\delta \in \mathbb{R}^{>0}$, $|r - s| \in \mu$.

Definition: Standard Part

For $r \in \mathbb{R}_{\text{fin}}$, the standard part of r is the unique $s \in \mathbb{R}$ such that $r \approx s$, written $\text{st}(r) = s$

Construction

Recall, $\mathbb{R}$ is the set of equivalence classes of Cauchy sequences of rational numbers. $\mathbb{R}^*$ can be constructed in a similar way. We say two sequences of real numbers are equivalent, denoted $(a_n) \sim (b_n)$ if $\{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{U}$ for some non principal ultrafilter $\mathcal{U}$. $\mathbb{R}^*$ is the set of sequences of real numbers with this equivalence relation.

Convergence of Sequences using $\mathbb{R}^*$

Convergence in $\mathbb{R}$

A sequence (s_n) in $\mathbb{R}$ converges to L if for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that for all $n \geq N$, then $|s_n - L| < \epsilon$

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Proposed extension:

Non-Standard Extension

A sequence $(s_n) \to L$ iff $s_N \approx L$ for $N > \mathbb{N}$

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Idea: $(s_n) \to L$ if and only if, for really large N, s_N is infinetly close to L.

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Proof: $\Rightarrow$ Suppose $(s_n) \to L$. Fix $\epsilon \in \mathbb{R}_{>0}$. Then, there is some fixed $m \in \mathbb{N}$, such that for all $n \in \mathbb{N}$

$$n \geq m \Rightarrow |s_n - L| < \epsilon.$$

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$$n \geq m \Rightarrow |s_n - L| < \epsilon.$$

Applying transfer principle, we get that for all $n \in \mathbb{N}^*$,

$$n \geq m \Rightarrow |s_n - L| < \epsilon$$

If $N > \mathbb{N}$, then $N > m$, so $|s_N - L| < \epsilon$, and $s_N \approx L$.

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$\Leftarrow$ Suppose $s_N \approx L$. Fix $\epsilon \in \mathbb{R}_{>0}$. Then, there is an $m > \mathbb{N}$ such that for

$$\text{all } n \in \mathbb{N}^* \; n \geq m \Rightarrow n > \mathbb{N} \text{ and } s_n \approx L$$

Therefore $|s_n - L| < \epsilon$.

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$$\text{all } n \in \mathbb{N}^* \ n \geq m \Rightarrow n > \mathbb{N} \text{ and } s_n \approx L$$

Therefore $|s_n - L| < \epsilon$. Applying the transfer principle, there is an $m \in \mathbb{N}$, such that for all $n \in \mathbb{N}$

$$n \geq m \Rightarrow |s_n - L| < \epsilon$$

□

Convergence of Sequences using $\mathbb{R}^*$

Example

Take $s_n = \frac{1}{n}$. Let $N > \mathbb{N}$.

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Then, for all $n \in \mathbb{N}$, $N > n$, so $0 < \frac{1}{N} < \frac{1}{n}$.

$\frac{1}{N} \in \mu$, so $\frac{1}{N} - 0 \in \mu$ and thus $\frac{1}{N} \approx 0$.

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Then, for all $n \in \mathbb{N}$, $N > n$, so $0 < \frac{1}{N} < \frac{1}{n}$.

$\frac{1}{N} \in \mu$, so $\frac{1}{N} - 0 \in \mu$ and thus $\frac{1}{N} \approx 0$.

Therefore, by our non-standard extension, since $\frac{1}{N} \approx 0$, $(s_n) \to 0$.

Note: No ϵ or limits required!

Convergence of Series using $\mathbb{R}^*$

Standard Definiton

A series $\sum_{n=0}^{\infty} a_n = L$ if and only if $\lim_{m\to\infty} S_m = L$ where $S_m = \sum_{n=0}^{m} a_n$.

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Proposed extension:

Non-Standard Extension

A series series $\sum_{n=0}^{\infty} a_n = L$ if and only if for $N > \mathbb{N}$, $\sum_{n=0}^{N} a_n \approx L$.

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A series series $\sum_{n=0}^{\infty} a_n = L$ if and only if for $N > \mathbb{N}$, $\sum_{n=0}^{N} a_n \approx L$.

Proof: Follows directly from the extension of sequences.

Convergence of Sequences using $\mathbb{R}^*$

Example

Consider $\sum_{n=0}^{\infty} r^n$ for $r \in (0, 1)$. Let $N > \mathbb{N}$.

After doing some algebra, we get that:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}.$$

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Since $r \in (0, 1)$, $r^{N+1} \approx 0$.

Therefore, $\sum_{n=0}^{N} r^n \approx \frac{1}{1-r}$. So $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

Note: Again no ϵ or limits required!

Series Theorem

Proposition

If $\sum_{n=0}^{\infty} a_n$ converges, then for $N > \mathbb{N}$, $a_N \approx 0$.

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Proof: Let $N > \mathbb{N}$ and $S_N = \sum_{n=0}^{N} a_n$. Then, $a_N = S_N - S_{N-1}$.

Since the series converges, $S_N \approx L$ and $S_{N-1} \approx L$.

Series Theorem

Proposition

If $\sum_{n=0}^{\infty} a_n$ converges, then for $N > \mathbb{N}$, $a_N \approx 0$.

Proof: Let $N > \mathbb{N}$ and $S_N = \sum_{n=0}^{N} a_n$. Then, $a_N = S_N - S_{N-1}$.

Since the series converges, $S_N \approx L$ and $S_{N-1} \approx L$.

So, $S_N - S_{N-1} \approx 0$. Therefore, $a_N \approx 0$.

□

Standard Continuity

Recall the ε-δ definition of continuity at a point.

Definition (Continuity at a Point)

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at $c \in \mathbb{R}$ if:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R},\ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

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Core idea: if x is very close to c, then $f(x)$ must be very close to $f(c)$.

- The "closeness" is controlled by δ and ε.
- A single δ works for a given c and ε.

Non-standard Continuity

Definition (Monad of a Point)

The **monad** of a standard real number c, denoted $\mu(c)$, is the set of hyperreal numbers infinitely close to c:

$$\mu(c) = \{x \in \mathbb{R}^* \mid x \approx c\}.$$

Non-standard Continuity

Idea

A function f is continuous at c if and only if it sends points infinitesimally close to c to points infinitesimally close to $f(c)$.

$$f(\mu(c)) \subseteq \mu(f(c)).$$

Simpler: $x \approx c$ **implies** $f(x) \approx f(c)$.

Non-standard vs. Standard Continuity

Standard (ε-δ)

- Requires a separate δ for each ε and each point c.
- Quantifiers over real numbers.

Nonstandard ($\approx$)

- A single condition: "x infinitely close to c implies $f(x)$ infinitely close to $f(c)$".
- Quantifiers over hyperreals.

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Theorem (Equivalence)

f is continuous at c in the standard sense if and only if for all $x \in \mathbb{R}^*$, $x \approx c$ implies $f(x) \approx f(c)$.

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Theorem (Equivalence)

f is continuous at c in the standard sense if and only if for all $x \in \mathbb{R}^*$, $x \approx c$ implies $f(x) \approx f(c)$.

This is a direct application of the Transfer Principle. Proofs are often reduced to algebraic manipulation with infinitesimals.

Example: Showing $f(x) = x^2$ is Continuous

Let's prove $f(x) = x^2$ is continuous at a standard point c using the nonstandard definition.

Nonstandard Proof.

Take any $x \in \mathbb{R}^*$ such that $x \approx c$.

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Take any $x \in \mathbb{R}^*$ such that $x \approx c$. This means $x = c + \varepsilon$, where $\varepsilon \in \mu$ is an infinitesimal.

Example: Showing $f(x) = x^2$ is Continuous

Let's prove $f(x) = x^2$ is continuous at a standard point c using the nonstandard definition.

Nonstandard Proof.

Take any $x \in \mathbb{R}^*$ such that $x \approx c$. This means $x = c + \varepsilon$, where $\varepsilon \in \mu$ is an infinitesimal. Now compute $f(x) - f(c)$:

$$
\begin{aligned}
f(x) - f(c) &= (c + \varepsilon)^2 - c^2 \\
&= c^2 + 2c\varepsilon + \varepsilon^2 - c^2 \\
&= 2c\varepsilon + \varepsilon^2.
\end{aligned}
$$

□

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So far:

$$x \approx c \implies f(x) - f(c) = 2c\varepsilon + \varepsilon^2$$

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Since c is a standard real number, $2c$ is finite. The product of a finite number and an infinitesimal is infinitesimal: $2c\varepsilon \in \mu$. Also, ε^2 is an infinitesimal.

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Since c is a standard real number, $2c$ is finite. The product of a finite number and an infinitesimal is infinitesimal: $2c\varepsilon \in \mu$. Also, ε^2 is an infinitesimal. The sum of two infinitesimals is infinitesimal. Hence $f(x) - f(c) \in \mu$, meaning $f(x) \approx f(c)$. This calculation is essentially the same as the standard proof but avoids the ε-δ management. □

Thank You!