Iteration of linear differential operators

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1. The original problem

Does iterating the derivative infinitely many times give a smooth function whenever it converges?

Asked 4 years, 6 months ago Modified 3 years, 6 months ago Viewed 5k times

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I am a graduate student and I've been thinking about this fun but frustrating problem for some time. Let $d = \frac{d}{dx}$, and let $f \in C^\infty(\mathbb{R})$ be such that for every real x,

$$g(x) := \lim_{n\to\infty} d^n f(x)$$

converges. A simple example for such an f would be $ce^x + h(x)$ for any constant c where $h(x)$ converges to 0 everywhere under this iteration (in fact my hunch is that every such f is of this form), eg. $h(x) = e^{x/2}$ or simply a polynomial, of course.

I've been trying to show that g is, in fact, differentiable, and thus is a fixed point of d. Whether this is true would provide many interesting properties from a dynamical systems point of view if one can generalize to arbitrary smooth linear differential operators, although they might be too good to be true.

Perhaps this is a known result? If so I would greatly appreciate a refere
has a trivial counterexample I've missed, please let me know. Otherwise
with some tricky double limit using tricks such as in [this MSE answer](), t

asked Jan 5, 2022 at 6:05

Paul Cusson

Any help is kindly appreciated.

1.1 The original question

Let $f \in C^\infty(\mathbb{R})$, and write

$$L_1 := \frac{\mathrm{d}}{\mathrm{d}x}.$$

Suppose that for every $x \in \mathbb{R}$ the pointwise limit

$$g(x) := \lim_{n \to \infty} L_1^n f(x)$$

exists.

Question. Must g be smooth? Moreover, can we prove $L_1 g = g$?

A direct answer:

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I was able to adapt the accepted answer to this MathOverflow post to positively answer the question. The point is that one can squeeze more out of Petrov's Baire category argument if one applies it to the "singular set" of the function, rather than to an interval.

The key step is to establish

> **Theorem 1**. Let $f \in C^\infty(\mathbf{R})$ be such that the quantity $M(x) := \sup_{m \geq 0} |f^{(m)}(x)|$ is finite for all x. Then f is the restriction to $\mathbf{R}$ of an entire function (or equivalently, f is real analytic with an infinite radius of convergence).

The convergence hypothesis actually forces f to be a restriction of an entire function.

answered Jan 6, 2022 at 4:06

Terry Tao

1.2 Tao's analyticity theorem

Theorem 1

Let $f \in C^\infty(\mathbb{R})$, and suppose that for every $x \in \mathbb{R}$,

$$M(x) := \sup_{m \geq 0} \left| f^{(m)}(x) \right| < \infty.$$

Then f is the restriction to $\mathbb{R}$ of an entire function. Equivalently, f is real analytic with infinite radius of convergence.

1.3 Why Tao's theorem applies

For each fixed $x \in \mathbb{R}$, the sequence

$$\left(f^{(n)}(x)\right)_{n=0}^{\infty}$$

converges, hence is bounded. Therefore

$$M(x) = \sup_{n \geq 0} \left| f^{(n)}(x) \right| < \infty,$$

so Tao's theorem applies immediately: f is the restriction of an entire function.

1.4 A locally uniform bound

Since f is the restriction of an entire function, expand $f^{(k)}$ at 0. Every coefficient is controlled by the *single* number $M(0)$:

$$f^{(k)}(x) = \sum_{n=0}^{\infty} \frac{f^{(n+k)}(0)}{n!} x^n,$$

$$\implies \left|f^{(k)}(x)\right| \leq \sum_{n=0}^{\infty} \frac{M(0)}{n!} |x|^n = M(0) e^{|x|}.$$

Note that for all $n \in \mathbb{N}$ and $x \in [a, b]$,

$$\left|f^{(n)}(x)\right| \leq C_{[a,b]} := \sup_{x \in [a,b]} M(0) e^{|x|} < \infty.$$

1.5 Take the limit on both sides

Fix $a < b$. For every n, the Fundamental Theorem of Calculus gives the identity

$$f^{(n)}(b) - f^{(n)}(a) = \int_a^b f^{(n+1)}(x)\,\mathrm{d}x$$

Taking $n \to \infty$ on both sides and applying LDCT to the right-hand side,

$$\begin{aligned} \lim_{n\to\infty}\left(f^{(n)}(b) - f^{(n)}(a)\right) &= \lim_{n\to\infty}\int_a^b f^{(n+1)}(x)\,\mathrm{d}x \\ \Longrightarrow \qquad g(b) - g(a) &= \int_a^b \lim_{n\to\infty} f^{(n+1)}(x)\,\mathrm{d}x \\ &= \int_a^b g(x)\,\mathrm{d}x. \end{aligned}$$

1.6 The limit is continuous

Since g is the pointwise limit of the derivatives and $\left|f^{(n)}(x)\right| \leq C_{[a,b]}$, we also have $|g(x)| \leq C_{[a,b]}$ on $[a, b]$. Therefore,

$$\begin{aligned} |g(b) - g(a)| &\leq \int_a^b |g(x)| \,\mathrm{d}x \\ &\leq C_{[a,b]}|b - a|. \end{aligned}$$

Thus g is locally Lipschitz, and hence continuous.

1.7 The limit satisfies $L_1 g = g$

Fix a. The integral identity becomes

$$g(x) = g(a) + \int_a^x g(t)\,\mathrm{d}t.$$

Since g is continuous, the Fundamental Theorem of Calculus yields

$$g'(x) = g(x).$$

Thus $L_1 g = g$, so g is smooth. Solving the ODE gives

$$g(x) = Ce^x$$

for some $C \in \mathbb{R}$.

2. Extensions

2.1 $L_2 f := af'$

$a \in C^\infty(\mathbb{R})$, with a nowhere zero

2.1.1 The fixed-point equation

Let $a \in C^\infty(\mathbb{R})$ be nowhere zero, and define

$$L_2 f := af'.$$

Suppose that the pointwise limit

$$g(x) := \lim_{n\to\infty} L_2^n f(x)$$

exists for every $x \in \mathbb{R}$.

We will show that:

$$g \text{ is smooth and } L_2 g = g$$

2.1.2 Change coordinates

Introduce the change of coordinates map

$$\Phi : \mathbb{R} \to \mathbb{R}, x \mapsto \int_0^x \frac{1}{a(t)} \, \mathrm{d}t.$$

It can be verified that Φ is a well-defined diffeomorphism from $\mathbb{R}$ onto its image. Moreover, $I := \Phi(\mathbb{R})$ is an open interval. Set

$$F := f \circ \Phi^{-1} : I \to \mathbb{R}.$$

The chain rule gives

$$F' = (L_2 f) \circ \Phi^{-1},$$

and by induction, for every $n \geq 0$,

$$F^{(n)} = (L_2^n f) \circ \Phi^{-1}.$$

2.1.3 Pull back the ordinary result

Note that the following limit exists for $y \in I$:

$$\lim_{n\to\infty} F^{(n)}(y) = \lim_{n\to\infty} (L_2^n f)\big(\Phi^{-1}(y)\big)$$
$$= g\big(\Phi^{-1}(y)\big)$$

So, we can define

$$G(y) := \lim_{n\to\infty} F^{(n)}(y).$$

The local ordinary derivative result on the interval I gives $G' = G$.

2.1.4 Recover the fixed-point equation

Since $G = g \circ \Phi^{-1}$, then $g = G \circ \Phi$. But $G = G'$, so g is smooth.

By the Fundamental Theorem of Calculus, $\Phi' = \frac{1}{a}$, so

$$\begin{aligned} g' &= (G' \circ \Phi) \cdot \Phi' \\ &= (G \circ \Phi) \cdot \frac{1}{a} \\ &= g \cdot \frac{1}{a}. \end{aligned}$$

Hence

$$\begin{aligned} L_2 g &= a g' \\ &= a\left(g \cdot \frac{1}{a}\right) \\ &= g. \end{aligned}$$

2.2 $L_3 f := af' + bf$

$a, b \in C^\infty(\mathbb{R})$, with a nowhere zero

2.2.1 Absorb the zeroth-order term

Let $a, b \in C^\infty(\mathbb{R})$, with a nowhere zero, and suppose

$$\lim_{n\to\infty} L_3^n f(x) = g(x)$$

for every $x \in \mathbb{R}$.

As in the af' case, define

$$\Phi(x) := \int_0^x \frac{1}{a(t)} \,\mathrm{d}t \quad (x \in \mathbb{R}), \quad I := \Phi(\mathbb{R}).$$

$$w(y) := \exp\left(\int_0^y (b \circ \Phi^{-1})(s) \,\mathrm{d}s\right) \quad (y \in I),$$

$$F := w \cdot \left(f \circ \Phi^{-1}\right) : I \to \mathbb{R}.$$

2.2.2 Iterate derivatives of H

Similarly, for $n \geq 0$,

$$F^{(n)} = w \cdot (L_3^n f) \circ \Phi^{-1}$$

Therefore, the pointwise limit

$$\begin{aligned} G(y) &:= \lim_{n\to\infty} F^{(n)}(y) \\ &= \lim_{n\to\infty} w(y) \cdot (L_3^n f)\big(\Phi^{-1}(y)\big) \\ &= w(y) \cdot g\big(\Phi^{-1}(y)\big). \end{aligned}$$

exists. As before, we have $G' = G$, so $L_3 g = g$. Moreover,

$$g = \frac{G \circ \Phi}{w \circ \Phi}$$

so g is smooth.

2.3 $L_4 f(x) := x f'(x)$

2.3.1 A simple zero

Suppose that for every $x \in \mathbb{R}$, the limit

$$g(x) := \lim_{n \to \infty} L_4^n f(x)$$

exists.

The rigidity of L_4 forces

$$f(x) = c_1 + c_2 x$$

for some $c_1, c_2 \in \mathbb{R}$. So,

$$g(x) = c_2 x$$

for all $x \in \mathbb{R}$ and

$$L_4 g = g.$$

2.4 $L_5 f(x) := x^2 f'(x)$

2.4.1 A second-order zero

Suppose that for every $x \in \mathbb{R}$, the limit $g(x) := \lim_{n\to\infty} L_5^n f(x)$ exists.

On each of $\mathbb{R}^+$ and $\mathbb{R}^-$, a change of coordinates turns L_5 into ordinary differentiation, so

$$g(x) = C_+ e^{-\frac{1}{x}} \quad (x > 0), \quad g(x) = C\_ e^{-\frac{1}{x}} \quad (x < 0).$$

One can show that $C\_$ is forced to be 0.

Therefore,

$$g(x) = \begin{cases} C_+ e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Thus, g is smooth and $L_5 g = g$.

3. Future directions

3.1 Beyond first-order differential operators

The guiding question is:

$$\lim_{n\to\infty} L^n f = g \text{ pointwise} \implies g \text{ is smooth and } Lg = g?$$

1. Higher-order ordinary differential operators.

$$Lf(x) = \sum_{k=0}^{m} a_k(x) f^{(k)}(x), \quad m \geq 2.$$

What do we do when the first-order change of coordinates is unavailable?

2. Partial differential operators.

$$Lf(x) = \sum_{|\alpha|\leq m} a_\alpha(x) \partial^\alpha f(x) \quad \text{on} \quad \mathbb{R}^d,$$

Questions?