Modular Forms

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Lagrange's 4-Square Thm

Problem: Can we write every natural number as a sum of 4 squares? That is, for $n \in \mathbb{N}$, does there exist some $a, b, c, d \in \mathbb{Z}$ so that

$$n = a^2 + b^2 + c^2 + d^2?$$

Idea: Let $r_4(n)$ be the number of ordered ways to write n as a sum of 4 squares. Create a generating series $\sum_{n=0}^{\infty} r_4(n) q^n$.

The θ Function

Start with a generating series $r_1(n)$, the number of ways to write n as a square.
We have $r_1(0) = 1$, $r_1(k^2) = 2$, and $r_1(n) = 0$ for n not a perfect square.

$$\theta(q) := \sum_{n=0}^{\infty} r_1(n) q^n = 1 + \sum_{k=1}^{\infty} 2q^{k^2}$$

Take the product θ^4:

$$\theta^4(q) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} r_1(a^2) r_1(b^2) r_1(c^2) r_1(d^2) q^{a^2+b^2+c^2+d^2}$$

Claim: This is the generating series we wanted.

Example: Coefficient of q^1

$$1 = 1^2 + 0^2 + 0^2 + 0^2 = \cdots = 0^2 + 0^2 + 0^2 + 1^2$$

So the number of ways to write 1 as an ordered sum of 4 squares is

$$r_1(1^2)r_1(0^2)r_1(0^2)r_1(0^2) + \cdots + r_1(0^2)r_1(0^2)r_1(0^2)r_1(1^2) = 8$$

Compare with

$$\theta^4(q) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty} r_1(a^2)r_1(b^2)r_1(c^2)r_1(d^2)q^{a^2+b^2+c^2+d^2}$$

So

$$\boxed{\theta^4(q) = \sum_{n=0}^{\infty} r_4(n)q^n}$$

Properties of θ^4

For $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$, let $q = e^{2\pi iz}$ in $\theta^4(q)$. As a function of z,

$$\theta^4(z) = \sum_{n=0}^{\infty} r_4(n) q^n = \sum_{n=0}^{\infty} r_4(n) e^{2\pi inz}$$

This is a Fourier series expansion of $\theta^4(z)$, and we can show it satisfies

$$\theta^4(z+1) = \theta^4(z) \tag{1}$$

$$\theta^4\left(\frac{z}{4z+1}\right) = (4z+1)^2\theta^4(z) \tag{2}$$

Eqn. (1) follows since $e^{2\pi iz}$ is periodic by 1.
Eqn. (2) is derived by Poisson summation.

Modular Forms

A **modular form of weight** $\mathbf{k}$ on $\Gamma \leq SL_2(\mathbb{Z})$ is a holomorphic function on the upper-half complex plane with the property that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

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Example: By its functional equations, θ^4 is a modular form of weight 2 on $\Gamma_0(4) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right\rangle$. For $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$,

$$\theta^4\left(\frac{1z+0}{4z+1}\right) = \theta^4\left(\frac{z}{4z+1}\right) \overset{\text{by eqn (2)}}{=} (4z+1)^2\theta^4(z)$$

Vector Space of Modular Forms

Theorem

Modular forms of weight 2 *on* $\Gamma_0(4)$ *form a complex vector space* $\mathcal{M}_2(\Gamma_0(4))$*.*

AHA! by a miracle we get that

Theorem

The dimension of $\mathcal{M}_2(\Gamma_0(4))$ *is 2.*

A basis consists of two well-understood functions $\{G_{2,2}(z), G_{2,4}(z)\}$. (These are called Eisenstein Series)

Definition of $G_{2,4}(z)$

$$G_{2,4}(z) = -\pi^2 \left(1 + 8 \sum_{n=1}^{\infty} \left(\sum_{0<d, d|n, d\notin 4\mathbb{Z}} d \right) e^{2\pi i z n} \right)$$

Notice we have a general formula to find each coefficient.

$$G_{2,4}(z) = -\pi^2 (1 + 8e^{2\pi i z} + 24e^{4\pi i z} + 32e^{6\pi i z} + \dots)$$

Proof of Lagrange's 4-Square Thm

Since $\theta^4(z) \in M_2(\Gamma_0(4))$ we can write

$$\theta^4(z) = aG_{2,4}(z) + bG_{4,4}$$

where through finite computations we get that

$$a = -\frac{1}{\pi^2} \quad \text{and} \quad b = 0$$

so then

$$\theta^4(z) = -\frac{1}{\pi^2} G_{2,4}(z)$$

Proof of Lagrange's 4-Square Thm

The n-th Fourier Coefficent of $\theta^4(z)$ is $r_4(n)$ and the n-th Fourier Coefficent for $G_{2,4}(z)$ is

$$-\pi^2 8 \sum_{0<d, d|n, d\notin 4\mathbb{Z}} d$$

So by comparing coefficents we get that

$$\boxed{r_4(n) = 8 \sum_{0<d, d|n, d\notin 4\mathbb{Z}} d}$$

Since $r_4(n) > 0$ for any n then n can be written as the sum of four squares. ☐

For the Curious...

We can use the θ function to solve similar problems.
For example, looking at the coefficient of θ^2:

$$\boxed{r_2(n) = 4 \sum_{0<d, d|n, d\notin 2\mathbb{Z}} (-1)^{(d-1)/2}}$$

This helps to prove

Theorem (Fermat's Sum of Two Squares)

For any prime p,there exists $x, y \in \mathbb{N}$ *such that* $p = x^2 + y^2$ *iff* $p \equiv 1 \pmod 4$

References

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Graduate Texts in Mathematics. Springer, 1973.

Thank you!