Orthogonal Polynomials and Random Matrices

Milagro Chen, John Kim
University of Waterloo, Ontario

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Supervised by Nathan Pagliaroli

Table of Contents

Problem Setup: Gaussian Matrix Integrals

- Let $\mathcal{H}_N = \{H \in M_N(\mathbb{C}) : H^\dagger = H\}$, and for an $H \in \mathcal{H}_N$, we define:
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- More specifically, our goal was to learn the tools to compute
$$\int_{\mathcal{H}_N} e^{-t\operatorname{Tr}(H^4)}\, d\mu(H).$$

Weyl's Integration Formula

Given a Unitary Invariant function, $f(H) = f(UHU^\dagger)$, the following holds:

$$\int_{\mathcal{H}_N} f(H)dH = C_N \int_{\mathbb{R}^N} f(UHU^\dagger) \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\lambda,$$

Where C_N is a normalizing factor.

Weyl's Integration Formula(Cont.)

Since the trace is Unitary Invariant function, $\mathrm{Tr}(H) = \mathrm{Tr}(UHU^\dagger)$, then we can apply W.I.F:

$$\int_{\mathcal{H}_N} e^{-\frac{1}{2}\mathrm{Tr}(H^2)} dH = C_N \int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N d\lambda_i$$

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$$= C_N \int_{\mathbb{R}^N} \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\mu(\lambda_i)$$

Vandermonde Determinant

Let's examine the 'Jacobian':

$$\prod_{1\le i<j\le N}(\lambda_i-\lambda_j)^2=\left[\det\begin{pmatrix}1 & 1 & \dots & 1\\ \lambda_1 & \lambda_2 & \dots & \lambda_N\\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_N^2\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1}\end{pmatrix}\right]^2,$$

which is called the 'Vandermonde Determinant'.

Vandermonde Determinant(Cont.)

Since we can multiply any row by a monic polynomial, let's multiply each ith row with the $i-1$th Hermite polynomial:

$$\prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 = \left[\det \begin{pmatrix} H_0(\lambda_1) & H_0(\lambda_2) & \dots & H_0(\lambda_N) \\ H_1(\lambda_1) & H_1(\lambda_2) & \dots & H_1(\lambda_N) \\ H_2(\lambda_1) & H_2(\lambda_2) & \dots & H_2(\lambda_N) \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-1}(\lambda_1) & H_{N-1}(\lambda_2) & \dots & H_{N-1}(\lambda_N) \end{pmatrix} \right]^2$$

This will allow us to compute the normalizing constant C_N :

$$C_N = \frac{1}{\prod_{k=1}^N k!}.$$

Table of Contents

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- The polynomials $H_0, H_1, \ldots, H_n$ form an orthogonal basis with respect to the inner product given by the standard Gaussian measure over the reals. This means $\langle H_n, H_m \rangle = 0$ whenever $n \neq m$; where
$$\langle P, Q \rangle = \int_{\mathbb{R}} PQ \, d\mu(\lambda).$$

Hermite Polynomials Examples

Here is the general form of the standard Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right) .$$

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Here are the first few in closed form:

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x.$$

Generalized Hermite Polynomials

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    - $\lambda P_k = P_{k+1} + R_k P_{k-1}$ for some constant $R_k = R_{V,k}(t)$.

Table of Contents

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- Using integration by parts, we derived the Discrete Painlevé I equation

$$k = R_k + 4tR_k(R_{k+1} + R_k + R_{k-1}).$$

Discrete Painlevé I Equation cont.

- Substituting t/N for t, dividing both parts by N, and setting $r_k(t) = \frac{R_k(t)}{N}$ we obtained

$$\frac{k}{N} = r_k(t) + 4t r_k(t)(r_{k+1}(t) + r_k(t) + r_{k-1}(t)).$$

Discrete Painlevé I Equation cont.

- Substituting t/N for t, dividing both parts by N, and setting $r_k(t) = \frac{R_k(t)}{N}$ we obtained

$$\frac{k}{N} = r_k(t) + 4tr_k(t)(r_{k+1}(t) + r_k(t) + r_{k-1}(t)).$$

- Taking the limit as k/N $\to x \in [0, 1]$ yields the tree equation:

$$x = r(1 + 12tr), \qquad \text{where} \qquad r = r(x, t) = \lim_{N\to\infty} \frac{R_{xN}(t)}{N}.$$

Solving the Matrix Integral

- Now we begin the work of solving the integral

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- Using Weyl's formula, we saw that this is equivalent to:

$$c_N \int_{\mathbb{R}^N} e^{-t(\lambda_1^4+\cdots+\lambda_N^4)} \prod_{i<j}(\lambda_i-\lambda_j)^2\,d\mu(\lambda_1)\cdots d\mu(\lambda_N),$$

with c_N as our normalizing constant.

Solving the Matrix Integral

- By expanding the determinant and integrating by parts over each of the N variables λ_k, we get

$$\int_{\mathcal{H}_N} e^{-t\ \mathrm{tr}(H^4)}\, d\mu(H) = c_N N!\, h_{N-1} h_{N-2} \cdots h_0.$$

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- Using our useful equation $h_k = R_k h_{k-1}$

$$\int_{\mathcal{H}_N} e^{-t\ \mathrm{tr}(H^4)}\, d\mu(H) = c_N N! R_1^{N-1} R_2^{N-2} \cdots R_{N-1} h_0.$$

Solving the Matrix Integral cont.

- Taking the log of the integral and dividing both sides by N^2, we get

$$\frac{1}{N^2}\log\int_{\mathcal{H}_N} e^{-\frac{t}{N}trH^4}d\mu(H) = -\frac{1}{N}\sum_{k=1}^{N}(1-\frac{k}{N})\log\frac{R_k(t)}{R_0(t)} - \log\frac{h_0(t)}{h_0(0)}.$$

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- The limit as $N\to\infty$ (and t being fixed) is

$$e_0(t) = -\int_0^1 (1-x)\log(\frac{r(x,t)}{x})dx.$$

Solving the Matrix Integral cont.

- Substituting $x = r(1 + 12tr)$ into our equation gives

$$e_0(t) = -\frac{1}{2}\log\alpha + \frac{1}{24}(1-\alpha)(\alpha-9)$$

where

$$\alpha(t) = r(1,t) = \frac{-1+\sqrt{1+48t}}{24t}$$

- $e_0(t)$ is called the free energy of the integral. It can give us valuable information about the model, and by taking the series expansion of $\alpha(t)$ we get a generating function that enumerates the plane maps with n vertices of degree 4

A Double-Trace Matrix Model

We now try to use this method to solve the integral

$$\int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr}(H^2)-\frac{Ng}{4}\operatorname{Tr}(H^4)-\frac{Nt}{2}(\operatorname{Tr}H^2)^2}\,dH.$$

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This matrix integral is more difficult because of the double trace term. In the large-N limit, factorization allows us to replace one occurrence of $\operatorname{Tr}(H^2)$ by its limiting second moment

$$m_2 = \lim_{N\to\infty}\frac{1}{N}\left\langle \operatorname{Tr}(H^2)\right\rangle.$$

The moments obtained from the matrix integral satisfy the corresponding Schwinger–Dyson equations, which form an infinite system of nonlinear recurrence relations among the moments of the matrix model.

A Double-Trace Matrix Model cont.

So substituting $\mathrm{Tr}(H^2)$ with m_2, we get

$$V(H) = \frac{1 + 2m_2 t}{2}\,\mathrm{Tr}(\mathrm{H}^2) + \frac{g}{4}\,\mathrm{Tr}(H^4).$$

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Solving for r gives us:

$$r(x,g,\omega) = \frac{-\omega + \sqrt{\omega^2 + 12gx}}{6g},$$

and we can also define

$$r(1,g,\omega) = \alpha(g,\omega) = \frac{-\omega + \sqrt{\omega^2 + 12g}}{6g}.$$

Finding the Free Energy

From the recursive orthogonal polynomial relation, and taking the large-N limit, we get

$$m_2 = 2\int_0^1 r(x)dx,$$

and simplify to get

$$\begin{aligned} 8t^3 m_2^4 + 12t^2 m_2^3 + (27g^2 + 36gt - 4t^2 + 6t)m_2^2 \\ +(18g - 4t + 1)m_2 - (16g + 1) = 0 \end{aligned}.$$

We can solve for m_2 in terms of g and t, and get an expression for the free energy

$$e_0(t) = -\frac{1}{2}\log(\alpha) - \frac{1}{24}\omega^2\alpha^2 + \frac{5}{12}\omega\alpha - \frac{3}{8}.$$

Table of Contents

Future Steps

- We have applied this approach to solve other multi-tracial matrix models, and have begun to create an outline to solve them for any general case.
- Calculate higher order moments using the Schwinger-Dyson Equations.
- Use these moments to determine the limiting eigenvalue distributions.
- We can find singularities of $e_0(t)$ to compute critical points and phase transitions of a system.

Thank you for listening!
Questions?