Normed Division Algebras and the Cayley–Dickson Construction

Motivation

We are familiar with the cross product in $\mathbb{R}^3$.

- Takes two vectors
- Produces a vector perpendicular to both
- Has useful geometric properties

Motivation

We are familiar with the cross product in $\mathbb{R}^3$.

- Takes two vectors
- Produces a vector perpendicular to both
- Has useful geometric properties

Natural question:

Can we define cross products in other dimensions?

To answer this, we study **normed division algebras**.

Normed Division Algebra

A **normed division algebra** A is an algebra over $\mathbb{R}$ satisfying

$$\|ab\| = \|a\| \, \|b\| \quad \text{for all } a, b \in A$$

where $\|a\|^2 = \langle a, a \rangle$
with standard Euclidean inner product

Normed Division Algebra

A **normed division algebra** A is an algebra over $\mathbb{R}$ satisfying

$$\|ab\| = \|a\| \, \|b\| \quad \text{for all } a, b \in A$$

where $\|a\|^2 = \langle a, a \rangle$
with standard Euclidean inner product
Examples:

- $\mathbb{R}$ (real numbers)
- $\mathbb{C}$ (complex numbers)

Real and Imaginary Parts

$$\dim(A) = n$$

- $\mathrm{Re}(A) = \{t1 : t \in \mathbb{R}\} \quad \rightarrow \quad \dim(\mathrm{Re(A)}) = 1$
- $\mathrm{Im}(A) = (\mathrm{Re}(A))^{\perp} \quad \rightarrow \quad \dim(\mathrm{Im(A)}) = n - 1$

Real and Imaginary Parts

$$\dim(A) = n$$

- $\mathrm{Re}(A) = \{t1 : t \in \mathbb{R}\} \;\rightarrow\; \dim(\mathrm{Re}(\mathrm{A})) = 1$
- $\mathrm{Im}(A) = (\mathrm{Re}(A))^{\perp} \;\rightarrow\; \dim(\mathrm{Im}(\mathrm{A})) = n - 1$

$a \in A$ has the following unique decomposition

$$a = \mathrm{Re}(a) + \mathrm{Im}(a)$$

where $\mathrm{Re}(a) \in \mathrm{Re}(A)$ and $\mathrm{Im}(a) \in \mathrm{Im}(A)$

Real and Imaginary Parts

The conjugate of a is the following

$$\overline{a} = \text{Re}(a) - \text{Im}(a)$$

Real and Imaginary Parts

The conjugate of a is the following

$$\overline{a} = \text{Re}(a) - \text{Im}(a)$$

$$\text{Re}(a) = \frac{1}{2}(a + \overline{a}) \ , \quad \text{Im}(a) = \frac{1}{2}(a - \overline{a})$$

Real and Imaginary Parts

The conjugate of a is the following

$$\overline{a} = \mathrm{Re}(a) - \mathrm{Im}(a)$$

$$\mathrm{Re}(a) = \frac{1}{2}(a + \overline{a}) \ , \quad \mathrm{Im}(a) = \frac{1}{2}(a - \overline{a})$$

Therefore

$$\overline{a} = -a \iff a \in \mathrm{Im}(A)$$

Real and Imaginary Parts

It can be shown that

$$\langle a, b \rangle = \mathrm{Re}(\bar{a}b) \qquad \|a\|^2 = a\bar{a} \quad \text{for all } a, b \in A$$

Real and Imaginary Parts

It can be shown that

$$\langle a, b\rangle = \mathrm{Re}(\bar{a}b) \qquad \|a\|^2 = a\bar{a} \quad \text{for all } a, b \in A$$

If $a, b \in \mathrm{Im}(A)$, we have $\bar{a} = -a$, so:

$$\langle a, b\rangle = \mathrm{Re}(-ab) = -\mathrm{Re}(ab) \qquad \mathrm{Re}(ab) = -\langle a, b\rangle$$

Real and Imaginary Parts

It can be shown that

$$\langle a, b\rangle = \mathrm{Re}(\bar{a}b) \qquad \|a\|^2 = a\bar{a} \quad \text{for all } a, b \in A$$

If $a, b \in \mathrm{Im}(A)$, we have $\bar{a} = -a$, so:

$$\langle a, b\rangle = \mathrm{Re}(-ab) = -\mathrm{Re}(ab) \qquad \mathrm{Re}(ab) = -\langle a, b\rangle$$

Decomposing ab into real and imaginary parts:

$$ab = -\langle a, b\rangle + \mathrm{Im}(ab)$$

Cross Product from Algebra

For $a, b \in \text{Im}(A)$ we define

$$a \times b = \text{Im}(ab)$$

Cross Product from Algebra

For $a, b \in \mathrm{Im}(A)$ we define

$$a \times b = \mathrm{Im}(ab)$$

This satisfies properties similar to the standard cross product:

- bilinear
- skew symmetric
- orthogonal to inputs

Cross Product from Algebra

For $a, b \in \mathrm{Im}(A)$ we define

$$a \times b = \mathrm{Im}(ab)$$

This satisfies properties similar to the standard cross product:

- bilinear
- skew symmetric
- orthogonal to inputs

$$\langle a \times b, a \rangle = 0$$

Norm Identity

For $a, b \in \text{Im}(A)$ we also have

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - \langle a, b \rangle^2$$

Norm Identity

For $a, b \in \mathrm{Im}(A)$ we also have

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - \langle a, b \rangle^2$$

This is exactly the same identity satisfied by the standard cross product in $\mathbb{R}^3$.

Norm Identity

For $a, b \in \mathrm{Im}(A)$ we also have

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - \langle a, b \rangle^2$$

This is exactly the same identity satisfied by the standard cross product in $\mathbb{R}^3$.

Thus normed division algebras naturally give rise to cross products.

Key Correspondence

There is a correspondence between:

- Normed division algebras
- Vector cross products

Key Correspondence

There is a correspondence between:

- Normed division algebras
- Vector cross products

Therefore understanding normed division algebras tells us:

In which dimensions cross products can exist.

Key Correspondence

There is a correspondence between:

- Normed division algebras
- Vector cross products

Therefore understanding normed division algebras tells us:

In which dimensions cross products can exist.

To construct these algebras we use the **Cayley–Dickson construction**.

Idea of the Construction

Given an algebra A, we construct a new algebra

$$A \oplus A$$

Idea of the Construction

Given an algebra A, we construct a new algebra

$$A \oplus A$$

Elements are pairs

$$(a, b)$$

with $a, b \in A$.

Idea of the Construction

Given an algebra A, we construct a new algebra

$$A \oplus A$$

Elements are pairs

$$(a, b)$$

with $a, b \in A$.
We define a special multiplication rule for these pairs.

Multiplication Rule

Multiplication is defined by

$$(a, b)(c, d) = (ac - \overline{d}b,\ da + b\overline{c})$$

Multiplication Rule

Multiplication is defined by

$$(a, b)(c, d) = (ac - \overline{d}b,\ da + b\overline{c})$$

This definition preserves key properties of the algebra structure.

Multiplication Rule

Multiplication is defined by

$$(a, b)(c, d) = (ac - \overline{d}b,\ da + b\overline{c})$$

This definition preserves key properties of the algebra structure.
It also allows us to define conjugation and a norm.

Conjugation and Norm

Define conjugation by

$$(a, b)^* = (\bar{a}, -b)$$

Conjugation and Norm

Define conjugation by

$$(a, b)^* = (\bar{a}, -b)$$

Define the norm

$$||(a, b)||^2 = ||a||^2 + ||b||^2$$

Conjugation and Norm

Define conjugation by

$$(a, b)^* = (\bar{a}, -b)$$

Define the norm

$$||(a, b)||^2 = ||a||^2 + ||b||^2$$

This keeps the multiplicative property of the norm.

Doubling Dimensions

Each application of the construction doubles the dimension.

Doubling Dimensions

Each application of the construction doubles the dimension.
Starting from $\mathbb{R}$ we obtain:

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$$

Doubling Dimensions

Each application of the construction doubles the dimension.
Starting from $\mathbb{R}$ we obtain:

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$$

Dimensions:

$$1 \to 2 \to 4 \to 8$$

Algebra Properties

As you continue with Cayley–Dickson doubling process, the algebras with higher dimensions gradually lose structure.

Algebra	Properties
$\mathbb{R}$	commutative, associative
$\mathbb{C}$	commutative, associative
$\mathbb{H}$	not commutative, associative
$\mathbb{O}$	not commutative, not associative (but alternative)

Beyond the Octonions

Algebra	Dimension	Normed Division Algebra?
$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$	1, 2, 4, 8	Yes
$\mathbb{S}$	16	No

The sedenions ($\mathbb{S}$) contain zero divisors.

Hurwitz Theorem

Hurwitz Theorem:

Hurwitz Theorem

Hurwitz Theorem:
The only real normed division algebras are

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}, \quad \mathbb{O}$$

Hurwitz Theorem

Hurwitz Theorem:
The only real normed division algebras are

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}, \quad \mathbb{O}$$

Their dimensions are

$$1, \quad 2, \quad 4, \quad 8$$

Consequence for Cross Products

Recall that cross products correspond to the imaginary parts of these algebras.

Consequence for Cross Products

Recall that cross products correspond to the imaginary parts of these algebras.
Dimensions of imaginary parts:

- $\mathbb{R}$: dimension 0
- $\mathbb{C}$: dimension 1
- $\mathbb{H}$: dimension 3
- $\mathbb{O}$: dimension 7

Consequence for Cross Products

Recall that cross products correspond to the imaginary parts of these algebras.
Dimensions of imaginary parts:

- $\mathbb{R}$: dimension 0
- $\mathbb{C}$: dimension 1
- $\mathbb{H}$: dimension 3
- $\mathbb{O}$: dimension 7

Therefore cross products exist only in

$$\mathbb{R}^3 \quad \text{and} \quad \mathbb{R}^7$$

Idea of the Proof

Sketch of proof:

- Start with a normed algebra B
- Build subalgebras using Cayley–Dickson

Idea of the Proof

Sketch of proof:

- Start with a normed algebra B
- Build subalgebras using Cayley–Dickson

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

Idea of the Proof

Sketch of proof:

- Start with a normed algebra B
- Build subalgebras using Cayley–Dickson

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

If we attempt to continue further, the resulting algebra is not alternative.

Idea of the Proof

Sketch of proof:

- Start with a normed algebra B
- Build subalgebras using Cayley–Dickson

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

If we attempt to continue further, the resulting algebra is not alternative.
Thus no additional normed division algebras exist.

Conclusion

- There are only **four normed division algebras**.
- Their dimensions are $1, 2, 4, 8$.
- This explains why cross products exist only in

$$\mathbb{R}^3 \quad \text{and} \quad \mathbb{R}^7$$

Conclusion

- There are only **four normed division algebras**.
- Their dimensions are $1, 2, 4, 8$.
- This explains why cross products exist only in

$$\mathbb{R}^3 \quad \text{and} \quad \mathbb{R}^7$$

A beautiful connection between algebra and geometry.

Thank You

Questions?