Calculus Meets Topology: Morse Theory

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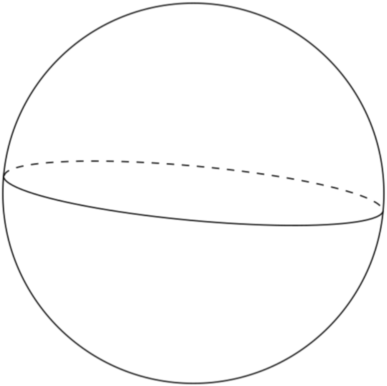
Motivation

1. How does a drop of water travel along a sphere?
   - "flow of water" is described by **Morse functions** and their critical points & "flows"

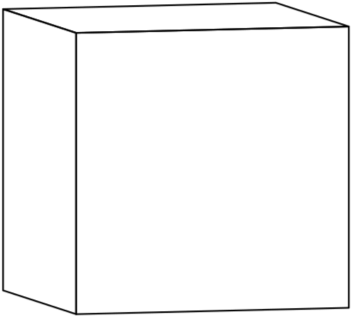
2. The flows cover the entire sphere $\Rightarrow$ they detect the shape of the sphere

3. "Shape" is described by **homology**, which can be retrieved from a Morse function on it, e.g. height function on a sphere
   - The homology we retrieve should not depend on the Morse function we choose

(Non)examples of Manifolds



The 3−dimensional sphere, S^2, is a smooth manifold (as is every S^n).



The 3−dimensional cube, however, is not a smooth manifold, as its vertices and edges are "too pointy."

Manifolds and Charts

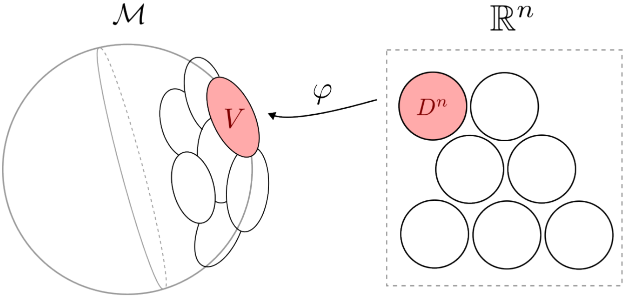
- A manifold is a **geometric shape:** locally, it looks flat
- A sphere is a manifold because it can be covered by a bunch of overlapping paper disks, without any sharp edges.
  - Each paper disk on the sphere is described using the notion of "charts":

> **Definition** A *chart* φ is a continuous bijection from an n-dimensional disk D^n to an open neighbourhood of $\mathcal{M}$.

- A manifold is a shape represented by a collection of local charts:

> **Definition** A topological space $\mathcal{M}$ is a *manifold* (of dimension n) if $\mathcal{M}$ is locally homeomorphic to the disk D^n. That is, there exist a collection of charts $S = \{\varphi : D^n \to V \subseteq \mathcal{M}\}$ such that
>
> $$\bigcup_{\varphi \in S} \varphi(D^n) = \mathcal{M}.$$

Example of a Manifold



- Because every neighbourhood V can be covered by a disc D^n, the sphere $\mathcal{M}$ is a manifold.
- For our purposes, manifolds are smooth

Morse Functions

- We can define (smooth) **functions** on manifolds
- We want functions whose "flows" detect the entire shape
  - Formally, we need the critical points to be non-degenerate.
  - Such functions are called **Morse functions,** and they satisfy the Morse Lemma:

Lemma (Morse Lemma:) *If* $f : \mathcal{M} \to \mathbb{R}$ *is a Morse function, then for every* $c \in \mathrm{Crit}(f)$*, there exists a chart* $\varphi : D^n \to \mathcal{M}$ *such that*

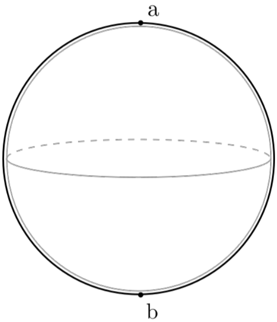
$$f \circ \varphi : D^n \to \mathbb{R}$$

is given by

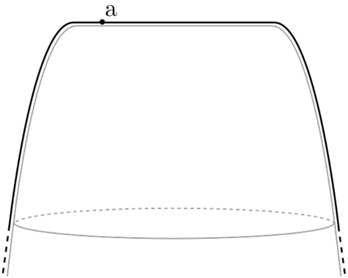
$$(x_1, \ldots, x_n) \mapsto f(c) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2.$$

Remark The value of k is dependent on c and is called the **index** of c.

A (Non)-example of a Morse Function



The height function on the sphere has only a maximum, a, and minimum, b, as critical points. Both are non-degenerate critical points.



The height function, on any manifold containing this "flat ridge" portion, is not a Morse function. The critical point a is degenerate.

Trajectories

How do we define the flow of a drop of water, mathematically?

Definition (Morse Trajectory:) A *Morse trajectory* is a map $\gamma : \mathbb{R} \to \mathcal{M}$ such that

$$\gamma'(t) = -\mathrm{Grad}_{\gamma(t)}(f)$$

Here, $\mathrm{Grad}_x(f)$ is the vector in the tangent space pointing in the direction of steepest increase of f at x.

Remark Assuming $\mathcal{M}$ is compact, the endpoints of γ given by $\gamma(\pm\infty) = \lim_{t\to\pm\infty} \gamma(t)$ are critical points of f.

Unstable manifold

Definition The *unstable manifold* of a critical point $c \in \text{Crit}(f)$ is

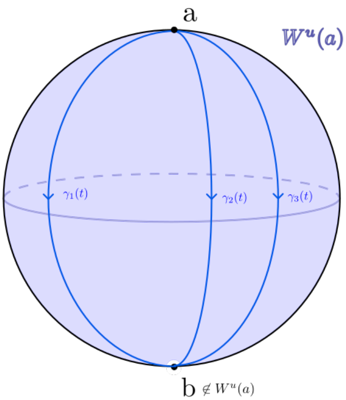
$$W^u(c) = \bigcup_{\substack{\gamma \text{ morse trajectory} \\ \gamma(-\infty)=c}} \{\gamma(t) : t \in \mathbb{R}\}$$

Definition The *stable manifold* of a critical point $c \in \text{Crit}(f)$ is

$$W^s(c) = \bigcup_{\substack{\gamma \text{ morse trajectory} \\ \gamma(\infty)=c}} \{\gamma(t) : t \in \mathbb{R}\}$$

- The unstable manifold describes all the possible flows downward from c, while the stable manifold describes all possible flows upward.

Unstable manifold in S^2



The unstable manifold of a, denoted $W^u(a)$, as well as some flows $\gamma_1, \gamma_2, \gamma_3$ in $W^u(a)$.

Note that b is not contained in $W^u(a)$, since any flow starting from a can not reach b in finite time.

What is Homology?

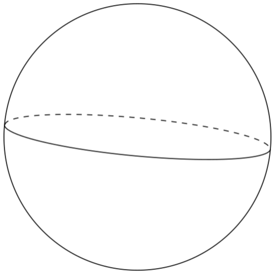
- Invariant under homotopy equivalence
- $H_0(X) \cong \mathbb{Z}$ if and only if X is path-connected
- Easier to compute than higher homotopy groups in general

Singular Homology Examples

- $H_0(S^n) = \mathbb{Z}$

  $H_n(S^n) = \mathbb{Z}$

  $H_m(S^n) = 0$ for all $m \neq n, 0$

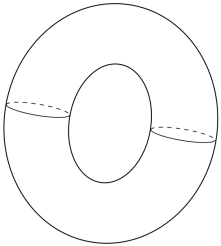


- $H_0(T^2) = \mathbb{Z}$

  $H_1(T^2) = \mathbb{Z}^2$

  $H_2(T^2) = \mathbb{Z}$

  $H_m(T^2) = 0$ for all $m \neq 0, 1, 2$



The Morse Complex

- A complex admits a chain of abelian groups and boundary maps.

Definition (Morse Complex:) Let $f : \mathcal{M} \to \mathbb{R}$ be a Morse function. Define $\mathrm{Crit}_k(f)$ to be the set of critical points with index k. Then for each k, we define an abelian group

$$C_k(f) = \left\{ \sum_{c \in \mathrm{Crit}_k(f)} a_c c \; : \; a_c \in \mathbb{Z} \right\}$$

where the a_c are signed, and a differential map

$$\partial^k : C_k \to C_{k-1}, \quad a \mapsto \sum_{b \in \mathrm{Crit}_{k-1}} n(a,b) \cdot b,$$

where $n(a,b)$ counts the (signed) number of trajectories from a to b. The (chain) complex

$$\cdots \to C_k \overset{\partial^k}{\to} C_{k-1} \overset{\partial^{k-1}}{\to} \ldots \overset{\partial^1}{\to} C_0$$

is called the *Morse complex.*

Morse Homology

Lemma *The Morse complex is a chain complex, i.e. that*

$$(\partial_{k-1} \circ \partial_k)(a) = \sum_{b \in \mathrm{Crit}_{k-2}} \left(\sum_{c \in \mathrm{Crit}_{k-1}} n(a,c) \cdot n(c,b) \right) b = 0$$

for all $a \in \mathrm{Crit}_k$.

Definition (The Morse Homology:) Let $f : \mathcal{M} \to \mathbb{R}$ be a Morse function, with critical points of index k contained in $\mathrm{Crit}_k(f)$. Then the k-th Morse homology group is

$$H_k(C_\star, f) = (\ker \partial_k)/(\mathrm{im}\, \partial_{k+1}).$$

Main Theorem

Theorem (Equivalence of Homologies:) *Let* $f, g : \mathcal{M} \to \mathbb{R}$ *be Morse functions. For each function, define Morse complexes* $C_\star(f), C_\star(g)$.

1. *For any* f, g, *then* $H_k(C_\star, f) = H_k(C_\star, g)$ *for each* k. *That is, the Morse homology does not depend on the function chosen.*
2. *For the simplicial complex* $C_\star^{\text{simp}}$ *and any Morse function* f, *then* $H_k(C_\star^{\text{simp}}) = H_k(C_\star, f)$ *for each* k. *That is, the Morse homology and singular homology are equivalent.*

We can then write $H_k(\mathcal{M})$ *to represent the homology of* $\mathcal{M}$.

Example: height function on 2-sphere

- Consider $f : S^2 \to \mathbb{R}$
- $\mathrm{Crit}_2(f) = \{a\}$, $\mathrm{Crit}_0(f) = \{b\}$, and $\mathrm{Crit}_n(f) = \varnothing$ for $n \neq 0, 2$
- The complex is $\cdots \to C_2 \overset{\partial^2}{\to} C_1 \overset{\partial^1}{\to} C_0 \overset{\partial^0}{\to} 0$
- Then $C_n(f)$ is trivial for $n \neq 0, 2$, and $C_0(f) = C_2(f) = \mathbb{Z}$, the free abelian group on one generator
- $\partial_n : C_n(f) \to C_{n-1}(f)$ is zero for all n
- $H_0^{\mathrm{Morse}}(S^2) = \ker \partial_0 / \mathrm{im}\, \partial_1 = \mathbb{Z}/0 = \mathbb{Z}$
- $H_2^{\mathrm{Morse}}(S^2) = \ker \partial_2 / \mathrm{im}\, \partial_3 = \mathbb{Z}/0 = \mathbb{Z}$
- $H_n^{\mathrm{Morse}}(S^2) = \ker \partial_n / \mathrm{im}\, \partial_{n+1} = 0/0 = 0$ for all $n \neq 0, 2$