Shifted Convolutions

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Classical additive divisor problem

Fix a positive integer k. Estimate the convolution

$$\sum_{n \leq X} d(n)d(n+k)$$

where $d(n)$ is the number of divisors of n.
How does shifting by an integer (addition) influence the number of divisors?

What are we estimating here?

A twisted additive divisor problem (Cowan, 2025):

$$\sum_{n<X} \frac{\sigma_{2u}(n,\chi)\sigma_{2v}(n+k,\psi)}{n^{u+v}}$$

where

$$\sigma_s(n,\chi) := \sum_{\substack{d|n \\ d>0}} \chi(d) d^s$$

What are we proving?

Assuming some mild hypotheses,

Theorem

$$\sum_{n=1}^{X} \frac{\sigma_{2u}(n,\chi)\sigma_{2v}(n-k,\psi)}{n^{u+v}} = \frac{L(1-2u,\chi)L(1-2v,\psi)}{L(2-2u-2v,\chi\psi)}\sigma_{-1+2u+2v}(k,\chi\psi)\frac{X^{1-u-v}}{1-u-v}$$
$$+\frac{\tau(\overline{\chi\psi})}{\tau(\bar{\chi})\tau(\bar{\psi})}\frac{L(1+2u,\bar{\chi})L(1+2v,\bar{\psi})}{L(2+2u+2v,\overline{\chi\psi})}\frac{\sigma_{1+2u+2v}(k,\chi\psi)}{k^{1+2u+2v}}\frac{X^{1+u+v}}{1+u+v}$$
$$+\mathcal{O}\left(X^{1+|\Re(u)|+|\Re(v)|-\frac{1+2|\Re(u)|+2|\Re(v)|}{3+|\Re(u+v)|+|\Re(u-v)|}+\varepsilon}\right)$$

There is a “main term” and an “error term”.

L-Series construction

Assume N is prime and some mild hypotheses. Define

$$V(z) := E^*_{\mathbb{1},\bar{\chi}}\left(z, \frac{1}{2} + u\right) E^*_{\mathbb{1},\bar{\psi}}\left(z, \frac{1}{2} + v\right)$$

where

$$E^*_{\mathbb{1},\bar{\psi}}(z,s) = \frac{N^s}{\tau(\psi)}\Lambda(2s,\bar{\psi})y^s + \sum_{n\neq 0} 2|n|^{\frac{1}{2}-s}\sigma_{2s-1}(n,\psi)y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)$$

is the completed Eisenstein series ([You19, Thm 4.1])

Proof idea

- Let $V(z) := E^*_{1,\overline{\chi}}\left(z, \frac{1}{2} + u\right) E^*_{1,\overline{\psi}}\left(z, \frac{1}{2} + v\right)$.
- Express $V(z)$ by two different methods. Calculate the Mellin transform of the kth Fourier coefficient of both forms of $V(z)$, then take an inverse Mellin transform (Perron's formula). Distinct expressions for the same thing gives identities for the shifted convolution.
- Method 1: Look up the Fourier series expansion of each E^* to calculate the Mellin transform of the kth Fourier coefficient directly.
- Method 2: Use a known basis (in terms of spectral decomposition) for square integrable functions.

Method 1

Lemma

$$\int_0^\infty \left(\int_0^1 V(z)\, e(-kx)\, dx \right) y^{s-1}\, dy$$

$$= \frac{\tau(\chi)}{\sqrt{N}} \Lambda(1+2u, \bar{\chi}) \frac{N^u \sigma_{2v}(k,\psi)}{2\pi^{s+u} k^{s+u+v}} \Gamma\left(\frac{s+u+v}{2}\right) \Gamma\left(\frac{s+u-v}{2}\right)$$

$$+ \frac{\tau(\psi)}{\sqrt{N}} \Lambda(1+2v, \bar{\psi}) \frac{N^v \sigma_{2u}(k,\chi)}{2\pi^{s+v} k^{s+u+v}} \Gamma\left(\frac{s+u+v}{2}\right) \Gamma\left(\frac{s-u+v}{2}\right)$$

$$+ \frac{\Gamma\left(\frac{s+u+v}{2}\right) \Gamma\left(\frac{s+u-v}{2}\right) \Gamma\left(\frac{s-u+v}{2}\right) \Gamma\left(\frac{s-u-v}{2}\right)}{2\pi^s \Gamma(s)}$$

$$\cdot \sum_{n \neq 0, k} \frac{\sigma_{2u}(n,\chi)\sigma_{2v}(n-k,\psi)}{|n|^{s+u+v}} \, {}_2F_1\left(\frac{s+u+v}{2}, \frac{s+u-v}{2}; s; \frac{2k}{n} - \frac{k^2}{n^2} \right)$$

Notation

$$L_k(s) := \sum_{n \neq 0, k} \frac{\sigma_{2u}(n,\chi)\sigma_{2v}(n-k,\psi)}{|n|^{s+u+v}} \, {}_2F_1\left(\frac{s+u+v}{2}, \frac{s+u-v}{2}; s; \frac{2k}{n} - \frac{k^2}{n^2} \right)$$

Method 2

For a square integrable U, we have its spectral decomposition in terms of Maass forms (spooky) and Eisenstein series,

$$U(z) = \sum_j \frac{\langle U, f_j \rangle}{\langle f_j, f_j \rangle} f_j(z) + \frac{1}{4\pi i} \sum_{\mathfrak{a}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle U, E_{\mathfrak{a}}(\cdot, \lambda, \chi\psi) \rangle E_{\mathfrak{a}}(z, \lambda, \chi\psi) d\lambda$$

Problem: $\langle V, E_{\mathfrak{a}}(\cdot, \lambda, \chi\psi) \rangle$ is not finite.

Regularizing function

If we choose R as a linear combination of two Eisenstein series

$$R(z) := \frac{\tau(\overline{\chi\psi})}{\tau(\overline{\chi})\tau(\overline{\psi})} \frac{\Lambda(1+2u,\overline{\chi})\Lambda(1+2v,\overline{\psi})}{\Lambda(2+2u+2v,\overline{\chi\psi})} E^*_{1,\overline{\chi\psi}}(z,1+u+v)$$
$$+ \frac{\Lambda(1-2u,\chi)\Lambda(1-2v,\psi)}{\Lambda(2-2u-2v,\chi\psi)} E^*_{1,\overline{\chi\psi}}(z,u+v).$$

then

Lemma

$V - R \in L^2(\Gamma_0(N)\backslash\mathbb{H};\chi\psi)$

Spectral decomposition of $V - R$

Then

$$V-R = \sum_j \frac{\langle V-R, f_j\rangle}{\langle f_j, f_j\rangle} f_j(z) + \frac{1}{4\pi i}\sum_{\mathfrak{a}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle V-R, E_{\mathfrak{a}}(\cdot,\lambda,\chi\psi)\rangle\, E_{\mathfrak{a}}(z,\lambda,\chi\psi)\,d\lambda$$

and we get

$$\begin{aligned}
&\int_0^\infty \left(\int_0^1 V(z)\, e(-kx)\, dx\right) y^{s-1}\,\frac{dy}{y} \\
&= \int_0^\infty \left(\int_0^1 (V(z)-R(z))\, e(-kx)\, dx\right) y^{s-1}\,\frac{dy}{y} \\
&\quad + \int_0^\infty \left(\int_0^1 R(z)\, e(-kx)\, dx\right) y^{s-1}\,\frac{dy}{y}
\end{aligned}$$

We calculate the contributions from the continuous spectrum and discrete spectrum separately.

Contribution from continuous spectrum

Lemma

$$\int_0^\infty \int_0^1 \left[\frac{1}{4\pi i}\sum_{\mathfrak{a}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle V-R, E_{\mathfrak{a}}(\cdot,\lambda,\chi\psi)\rangle E_{\mathfrak{a}}(z,\lambda,\chi\psi)\, d\lambda\right] e(-kx)y^{s-1}\frac{dx\,dy}{y}$$

$$= \int_{-\infty}^{\infty} \frac{\Lambda(\frac{1}{2}+i\omega+u+v)\Lambda(\frac{1}{2}+i\omega-u+v,\chi)\Lambda(\frac{1}{2}+i\omega+u-v,\psi)\Lambda(\frac{1}{2}+i\omega-u-v,\chi\psi)}{\Lambda(1+2i\omega,\chi\psi)\Lambda(1-2i\omega,\chi\psi)}$$

$$\cdot\frac{\tau(\chi\psi)}{\sqrt{N}}\frac{\sigma_{-2i\omega}(k,\chi\psi)}{\left(\frac{\frac{1}{2}-i\omega-u-v}{2}\right)\pi^{s+\frac{1}{2}}k^{s-\frac{1}{2}-i\omega}}\Gamma\left(\frac{s-\frac{1}{2}+i\omega}{2}\right)\Gamma\left(\frac{s-\frac{1}{2}-i\omega}{2}\right)d\omega.$$

Contribution from discrete spectrum

Lemma

$$\int_0^\infty \int_0^1 \left[\sum_j \frac{\langle V - R, f_j \rangle}{\langle f_j, f_j \rangle} f_j(z) \right] e(-kx) y^{s-1} \frac{dx\,dy}{y}$$

$$= \sum_{f_j \text{ even}} \frac{\tau(\chi)}{\sqrt{N}} \frac{N^u |\rho_j(1)|^2 a_j(k)}{2\pi^{s+\frac{1}{2}+2u} k^{s-\frac{1}{2}} \langle f_j, f_j \rangle} L(\tfrac{1}{2} + u + v, \overline{f_j}) L(\tfrac{1}{2} + u - v, \psi \times \overline{f_j})$$

$$\cdot \Gamma\left(\frac{u + v + \frac{1}{2} + ir_j}{2} \right) \Gamma\left(\frac{u - v + \frac{1}{2} + ir_j}{2} \right) \Gamma\left(\frac{u + v + \frac{1}{2} - ir_j}{2} \right) \Gamma\left(\frac{u - v + \frac{1}{2} - ir_j}{2} \right)$$

$$\cdot \Gamma\left(\frac{s - \frac{1}{2} + ir_j}{2} \right) \Gamma\left(\frac{s - \frac{1}{2} - ir_j}{2} \right).$$

Contribution from regularizing function

Lemma

$$\int_0^\infty \left(\int_0^1 R(z)\, e(-kx)\, dx \right) y^{s-1} \frac{dy}{y}$$

$$= \frac{\tau(\overline{\chi\psi})}{\tau(\overline{\chi})\tau(\overline{\psi})} \frac{\Lambda(1+2u,\overline{\chi})\Lambda(1+2v,\overline{\psi})\sigma_{2u+2v+1}(k,\chi\psi)}{\Lambda(2+2u+2v,\overline{\chi\psi})} \frac{\Gamma\left(\frac{s+u+v}{2}\right)}{\Gamma\left(\frac{s-1+u-v}{2}\right)}$$

$$+ \frac{\Lambda(1-2u,\chi)\Lambda(1-2v,\psi)\sigma_{2u+2v-1}(k,\chi\psi)}{\Lambda(2-2u-2v,\chi\psi)} \frac{\Gamma\left(\frac{s-u-v}{2}\right)}{\Gamma\left(\frac{s-1+u-v}{2}\right)}$$

Putting all the terms together

We write

$$L_k(s) = L_k^{(\mathrm{cont})}(s) + L_k^{(\mathrm{disc})}(s) + L_k^{(R)}(s) + L_k^{(V)}(s)$$

where $L_k^{(\cdot)}(s)$ are results from the last few slides multiplied by the constant term

$$\frac{2\pi^s\Gamma(s)}{\Gamma\left(\frac{s+u+v}{2}\right)\Gamma\left(\frac{s+u-v}{2}\right)\Gamma\left(\frac{s-u+v}{2}\right)\Gamma\left(\frac{s-u-v}{2}\right)}$$

Perron's Formula

Lemma (Perron)

$$\int_{1+|R(u)|+|R(v)|+\epsilon+iT}^{1+|R(u)|+|R(v)|+\epsilon-iT} L_k^*(s)X^s\frac{ds}{s} = 4\pi i\sum_{n=1}^{X}\frac{\sigma_{2u}(N,\chi)\sigma_{2v}(n-k,\psi)}{n^s} + \mathcal{O}\left(\frac{X^{1+|R(u)|+|R(v)|+\epsilon}}{T}\right)$$

Lemma (contour integration)

$$\int_{1+|\Re(u)|+|\Re(v)|+\varepsilon-iT}^{1+|\Re(u)|+|\Re(v)|+\varepsilon+iT} L_k^*(s)X^s\frac{ds}{s} = \sum_{\sigma>\frac{1}{2}+\varepsilon} 2\pi i\operatorname{Res}_s L_k(s)\frac{X^s}{s} + \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} L_k(s)X^s\frac{ds}{s}$$

$$-\left(\int_{\frac{1}{2}+\varepsilon-iT}^{1+|\Re(u)|+|\Re(v)|+\varepsilon-iT} - \int_{\frac{1}{2}+\varepsilon+iT}^{1+|\Re(u)|+|\Re(v)|+\varepsilon+iT}\right) L_k^*(s)X^s\frac{ds}{s} + \mathcal{O}\left(X^{\frac{1}{2}+\varepsilon}T^{\frac{1}{2}+\varepsilon}\right).$$

The main term comes from $\sum_{\sigma>\frac{1}{2}+\varepsilon} 2\pi i\operatorname{Res}_s L_k(s)\frac{X^s}{s}$

Final Step

Lemma

$$\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} L_k^{(R)}(s)X^s\frac{ds}{s} \ll X^{|\Re(u+v)|+\varepsilon} + X^{\varepsilon}T^{\varepsilon} + X^{\frac{1}{2}+\varepsilon}T^{-1}.$$

$$\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} L_k^{(V)}(s)X^s\frac{ds}{s} \ll X^{|\Re(u-v)|+\varepsilon} + X^{-\Re(u+v)+\varepsilon} + X^{\varepsilon}\left(T^{\frac{1}{2}+\Re(u)} + T^{\frac{1}{2}+\Re(v)}\right) + X^{\frac{1}{2}+\varepsilon}\left(T^{-\frac{1}{2}+\Re(u)} + T^{-\frac{1}{2}+\Re(v)}\right).$$

$$\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} L_k^{(\mathrm{cont})}(s)X^s\frac{ds}{s} \ll \left(X^{\frac{1}{2}}T^{\frac{1}{2}} + X^{\varepsilon}T + X^{\frac{1}{2}+\varepsilon}\right)T^{\varepsilon}.$$

$$\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} L_k^{(\mathrm{disc})}(s)X^s\frac{ds}{s} \ll \left(X^{\frac{1}{2}}T^{\frac{1}{2}} + X^{\varepsilon}T + X^{\frac{1}{2}+\varepsilon}\right)T^{\frac{|\Re(u+v)|+|\Re(u-v)|}{2}+\varepsilon}.$$

$$\left(\int_{\frac{1}{2}+\varepsilon-iT}^{1+|\Re(u)|+|\Re(v)|+\varepsilon-iT} - \int_{\frac{1}{2}+\varepsilon+iT}^{1+|\Re(u)|+|\Re(v)|+\varepsilon+iT}\right) L_k^*(s)X^s\frac{ds}{s} \ll X^{\frac{1}{2}+\varepsilon}T^{\frac{|\Re(u+v)|+|\Re(u-v)|}{2}+\varepsilon} + X^{1+|\Re(u)|+|\Re(v)|+\varepsilon}T^{-1}.$$

Balancing the error terms gives $\mathcal{O}\left(X^{1+|\Re(u)|+|\Re(v)|-\frac{1+2|\Re(u)|+2|\Re(v)|}{3+|\Re(u+v)|+|\Re(u-v)|}+\varepsilon}\right)$

A generalization?

In this problem we defined

$$V(z) := E^*_{\mathbb{1},\bar{\chi}}\left(z, \frac{1}{2}+u\right) E^*_{\mathbb{1},\bar{\psi}}\left(z, \frac{1}{2}+v\right)$$

Consider

$$V(z) = y^{\frac{k}{2}} E^*_{\chi_1,\chi_2}\left(z, \frac{k}{2}\right) \overline{E^*_{\psi_1,\psi_2}\left(z, \frac{k}{2}\right)}$$

(Holomorphic Eisenstein Series)

Work in progress...

Thanks for listening!

Figure: Alex