Why the Golden ratio appears in Plant Patterns

Mentor: **Jérémy Champagne**
Mentees: **Jackie Liu**, **Lydia He**

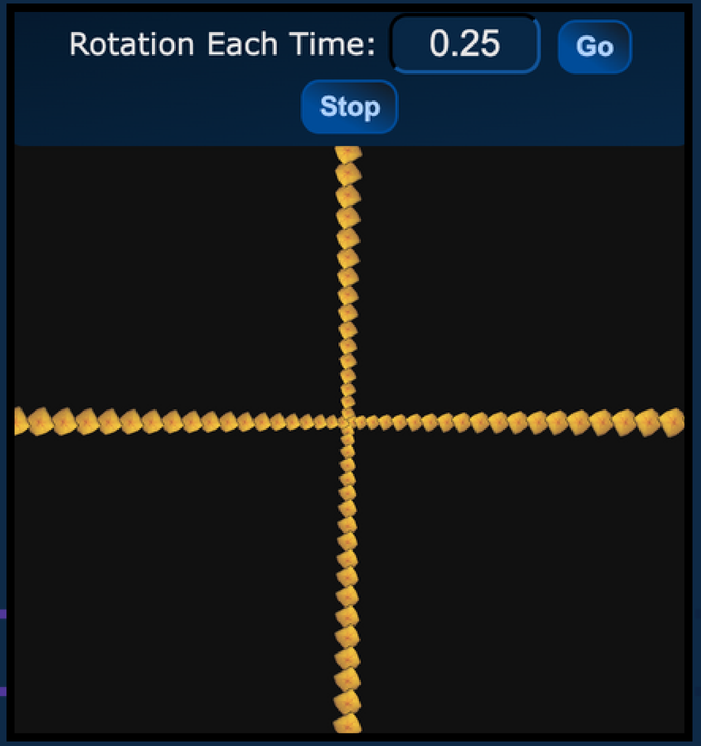
If you were a plant that grew in spirals...

Then you have limited space to plant your seeds/grow your leaves.

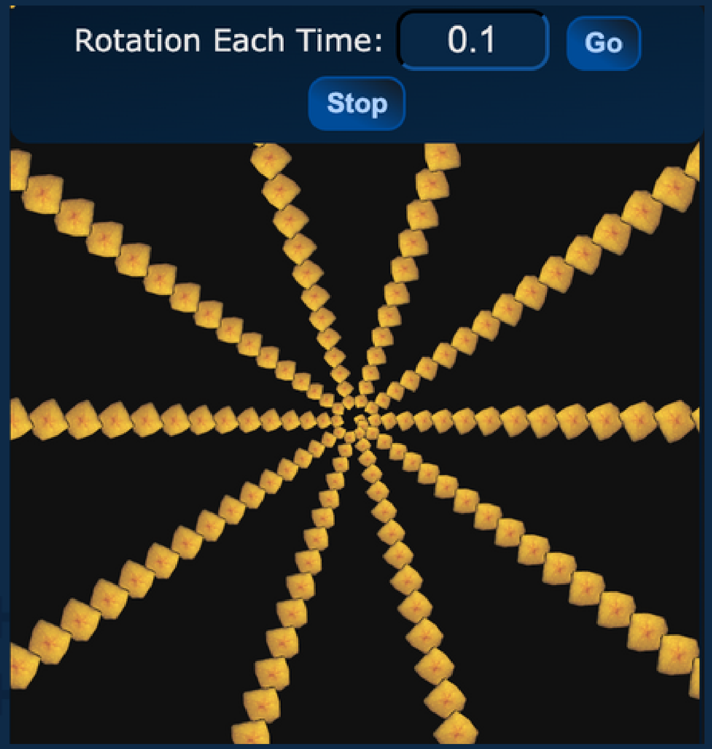
How much would you turn between each consecutive seed to maximize the number of seeds you grow?

A Model for Growth of a Sunflower (√n, nθ)

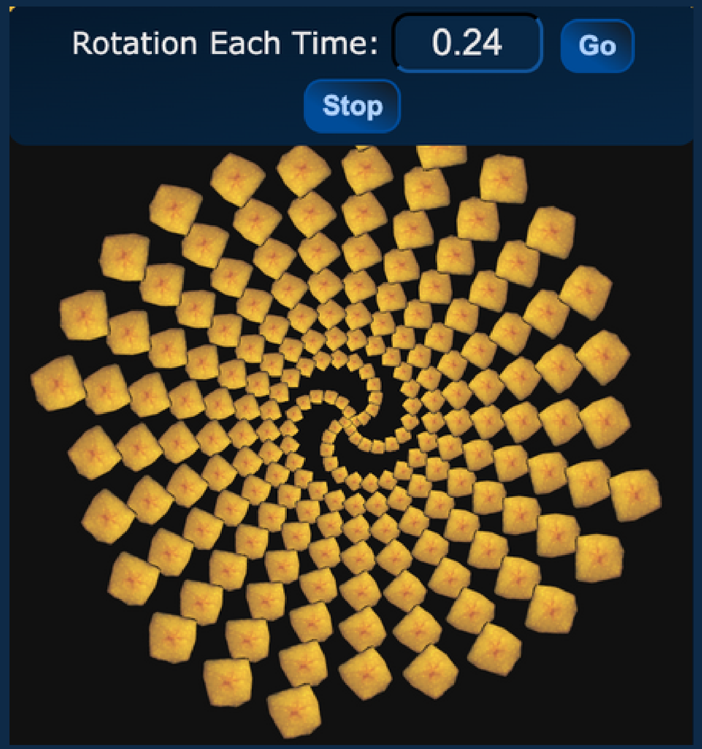
- An accepted model for sunflower growth is as follows: the n-th seed is placed at angle nθ and distance √n from the center, for some fixed θ.
- In these examples, we observe the changes that occur when θ takes on different values.
- Note that the model is very inefficient when θ is a rational number with a small denominator.
- Spirals become apparent when θ is “badly approximable”
- We wish for θ to be the "farthest" possible from a rational number. *But what does that mean?*



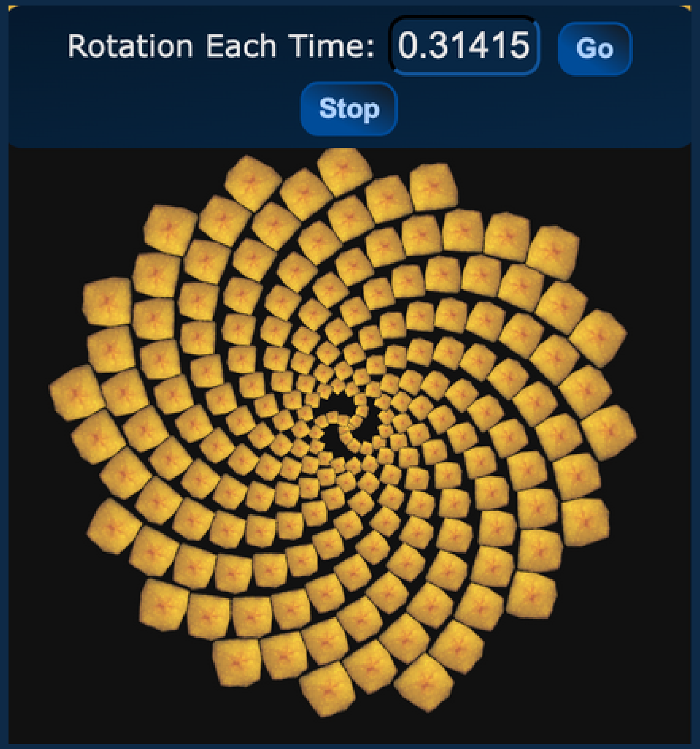
Rotation: **1/4**



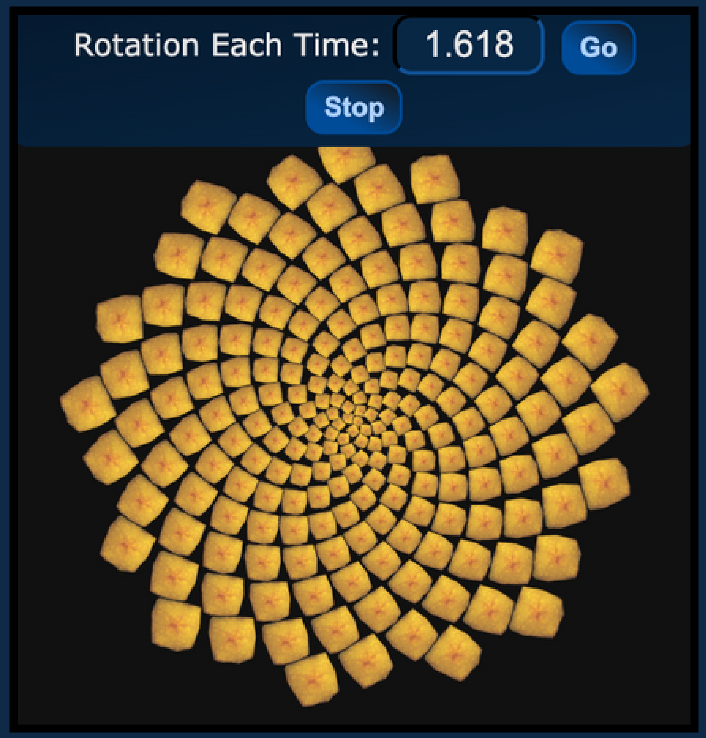
Rotation: **1/10**



Rotation: **0.24**



Rotation: **π/10**



Rotation: **1.618 = φ**

Diophantine Approximations

What is a “good” approximation?

We say a rational $\frac{p}{q}$ is a ”good” approximation of α if $\left|\alpha - \frac{p}{q}\right|$ is negligible for a relatively small denominator q.

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Example: √2

The first few approximations p, q for $\left|\sqrt{2} - \frac{p}{q}\right|$:

$$|\sqrt{2} - \tfrac{1}{1}| = 0.4142135$$

$$|\sqrt{2} - \tfrac{3}{2}| = 0.0857864$$

$$|\sqrt{2} - \tfrac{7}{5}| = 0.0142135$$

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Dirichlet's approximation theorem

For all $\alpha \in \mathbb{Q}$, there exist ∞ many pairs of integers p, q

$$\text{for which } \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

$$< 1$$

$$< \frac{1}{2^2}$$

$$< \frac{1}{5^2}$$

$$< \frac{1}{12^2}$$

Approximations for √2

A few observations...

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots \sqrt{2}$$

Observation 1: Recurrence relation

$$\begin{bmatrix}7\\5\end{bmatrix} = 2\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}17\\12\end{bmatrix} = 2\begin{bmatrix}7\\5\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix}$$

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Observation 2: The determinant is ±1

$$\det\left(\begin{bmatrix}1 & 3\\1 & 2\end{bmatrix}\right) = -1$$

$$\det\left(\begin{bmatrix}3 & 7\\2 & 5\end{bmatrix}\right) = 1$$

$$\det\left(\begin{bmatrix}7 & 17\\5 & 12\end{bmatrix}\right) = -1$$

These observations will be formalized as properties of continued fractions.

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Continued Fractions

Defining a Continued Fraction

Finite

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

$$\alpha = [a_0; a_1, a_2, ..., a_n]$$

Infinite

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

$$\alpha = [a_0; a_1, a_2, ...]$$

We will only focus on $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$

Facts:

1. Finite continued fraction $\iff$ α is rational
2. To every real number α, there corresponds a **unique** continued fraction with value equal to α

Convergents & Properties

The kth-order convergent of α:

$$\frac{p_k}{q_k} = [a_0; a_1, a_2, ..., a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

Then, $\frac{p_0}{q_0} = \frac{a_0}{1}$. We also consider the -1 order convergent:

$$p_{-1} = 1,\ q_{-1} = 0,\ p_0 = a_0,\ q_0 = 1$$

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Property 1

For arbitrary $k \geq 1$,

$$p_k = a_k p_{k-1} + p_{k-2},$$
$$q_k = a_k q_{k-1} + q_{k-2}.$$

Property 2

For all $k \geq 0$,

$$\det\left(\begin{bmatrix} q_k & q_{k-1} \\ p_k & p_{k-1} \end{bmatrix}\right) = (-1)^k$$

Property 3

For arbitrary $k \geq 0$,

$$\frac{1}{2q_k q_{k+1}} < \left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}}$$

Representing the Golden Ratio

The quadratic equation...

$$x^2 - x - 1 = 0$$

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$$x^2 - x - 1 = 0$$

The **golden ratio** is the positive root

$$\varphi = \frac{1+\sqrt{5}}{2}$$

Rearranging the equation...

$$x = 1 + \frac{1}{x}$$

Continued fraction of the golden ratio:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} = [1; 1, 1, ...]$$

A Relation to Fibonacci

Definition of the **Fibonacci sequence**:

Let $\{F_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be the Fibonacci sequence, that is:

$$F_0 = F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{Z}_{\geq 0})$$

Convergents of the **Golden ratio**:

$$\frac{p_0}{q_0} = \frac{1}{1} = \frac{F_1}{F_0}$$

$$\frac{p_1}{q_1} = 1 + \frac{1}{1} = \frac{2}{1} = \frac{F_2}{F_1}$$

$$\frac{p_2}{q_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_1 + p_0}{q_1 + q_0} = \frac{3}{2} = \frac{F_3}{F_2}$$

$$\frac{p_3}{q_3} = \frac{a_3 p_2 + p_1}{a_3 q_2 + q_1} = \frac{p_2 + p_1}{q_2 + q_1} = \frac{5}{3} = \frac{F_4}{F_3}$$

$$\frac{p_4}{q_4} = \frac{a_4 p_3 + p_2}{a_4 q_3 + q_2} = \frac{p_3 + p_2}{q_3 + q_2} = \frac{8}{5} = \frac{F_5}{F_4}$$

$$\cdots$$

$$\frac{p_n}{q_n} = \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} = \frac{F_n + F_{n-1}}{F_{n-1} + F_{n-2}} = \frac{F_{n+1}}{F_n}$$

Fibonacci Numbers!

Is there a C > 0, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^2}$$

has **infinitely** many solutions $\frac{p}{q}$?

1. α is irrational, can always choose C $= \frac{1}{\sqrt{5}}$.

2. α is in the form $\frac{a\varphi + b}{c\varphi + d} = [a_0; a_1, ..., a_n, \overline{1, ..., 1}]$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc = \pm 1$ and φ is the golden ratio, then C cannot be less than $\frac{1}{\sqrt{5}}$. Otherwise, C $= \frac{1}{\sqrt{8}}$ works.

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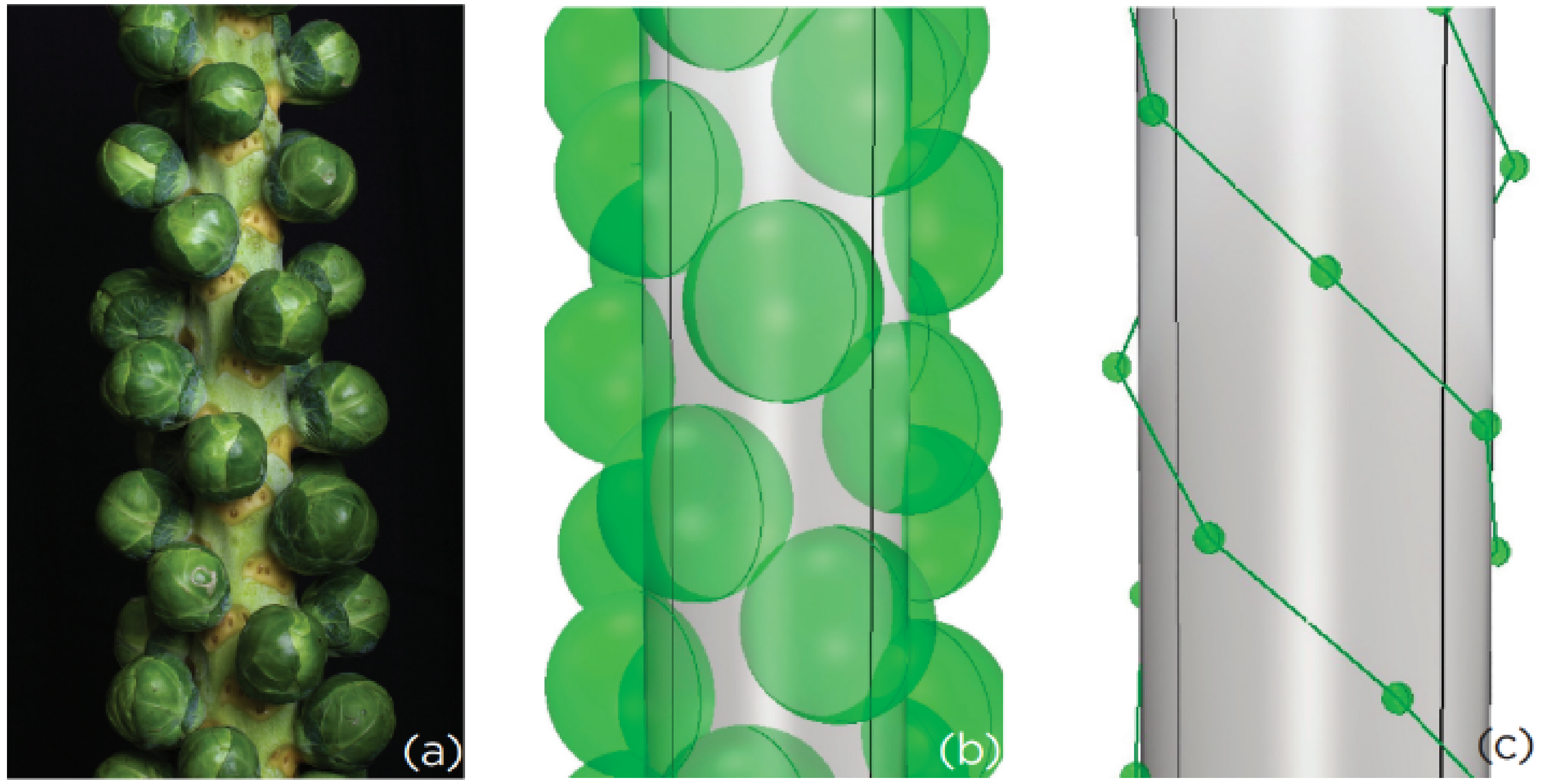
Fibonacci Spirals

Brussels Sprouts

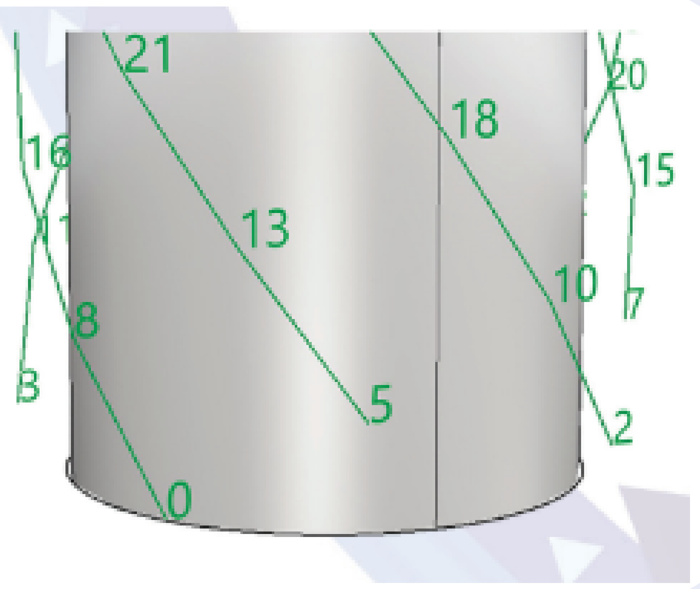


- A cylinder of circumference 1
- Add points by:
  - Placing the first point at the bottom.
  - Rotating by 360/φ degrees and moving up a vertical distance h, place the second point
  - Repeating ...

- The apparent spirals within the points form as each point connects to its two closest neighbours.

- If point n is the closest to 0, then there will be n apparent spirals.
  - same distance from 0 to n, n to 2n, ...
  - same distance from 0 to n, 1 to n+1, ..., n-1 to 2n-1

- The number of apparent spirals is a denominator in a convergent of the rotation (360/φ), which will be a Fibonacci number.

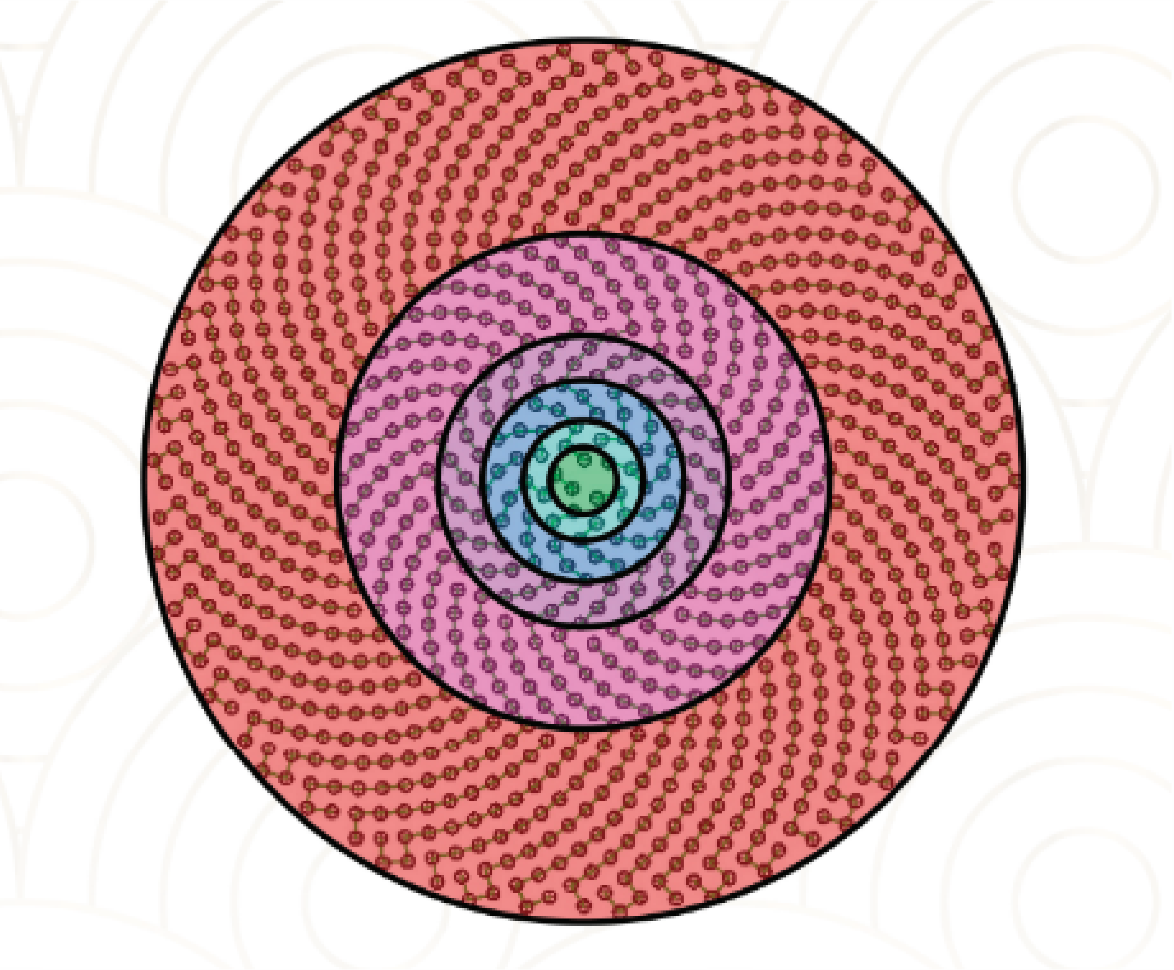


A model with 8 spirals

Sunflower

- The number of spirals is also always a Fibonacci number.
- Farther from the center, the numbers of apparent spirals increase and they alternate directions.

Red: 34 spirals **Blue**: 55 spirals

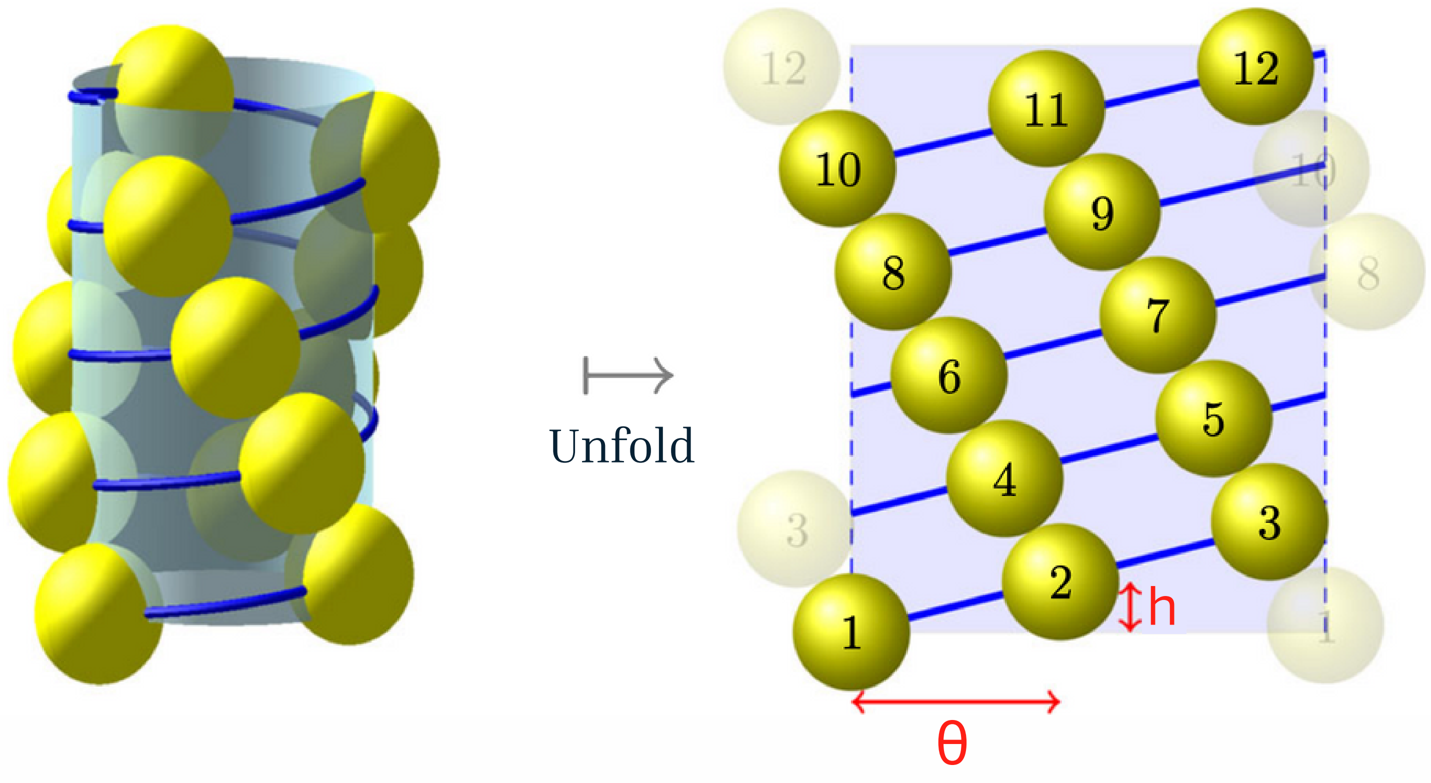


04

Golden ratio and phyllotaxis:
a clear mathematical link

**Phyllotaxis* is the study of plant patterns

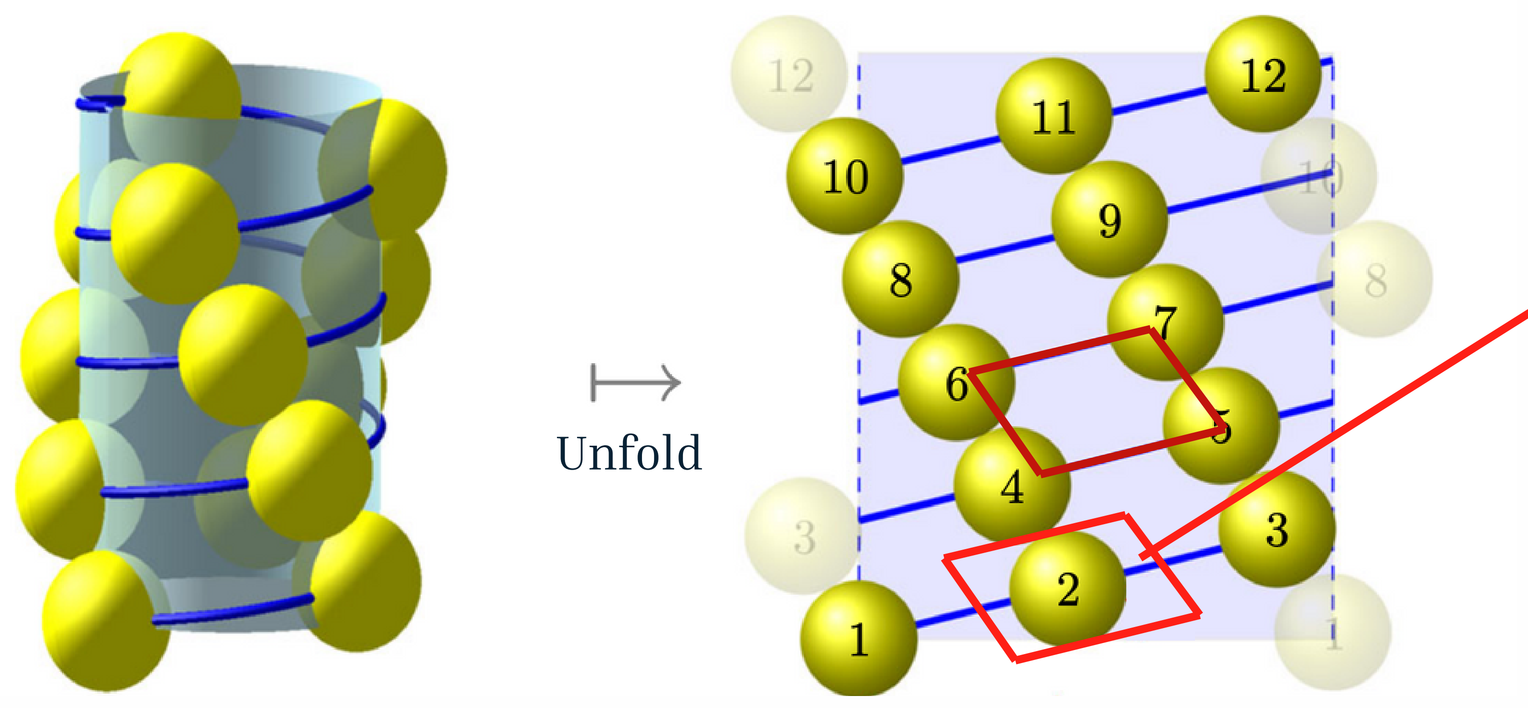
The Lattice Model



Buds on a cylindrical stem

A lattice in $\mathbb{C}$

The Lattice Model



Buds on a cylindrical stem

A lattice in ℂ

Draw a parallelogram around one of these buds...

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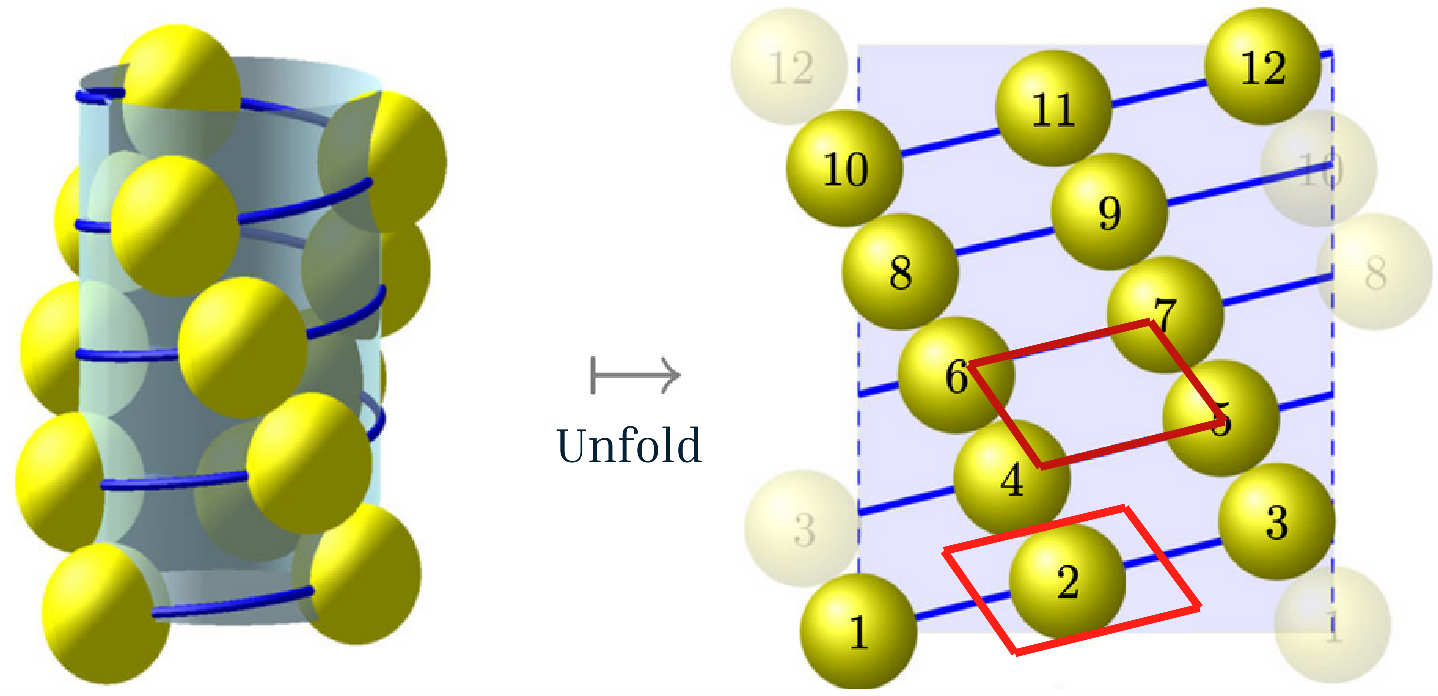

Buds on a cylindrical stem

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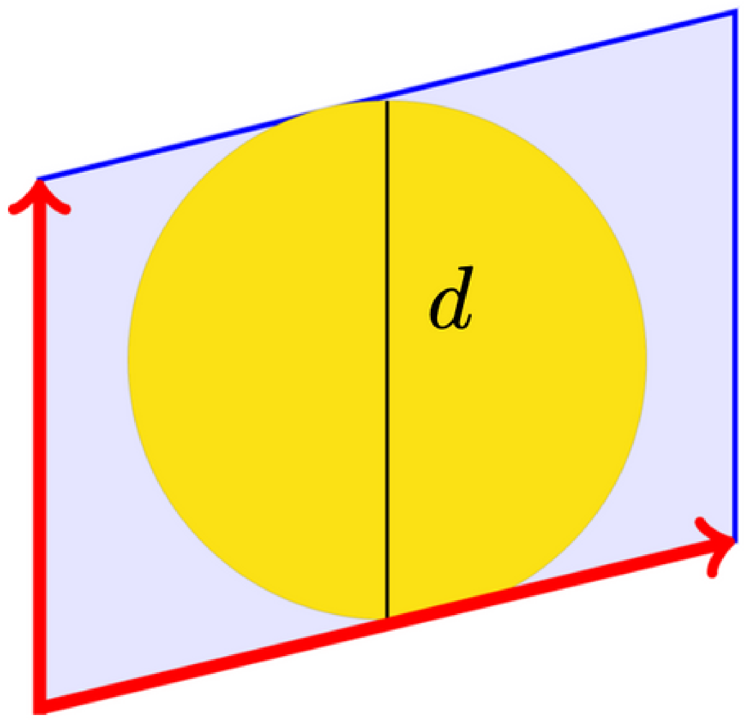
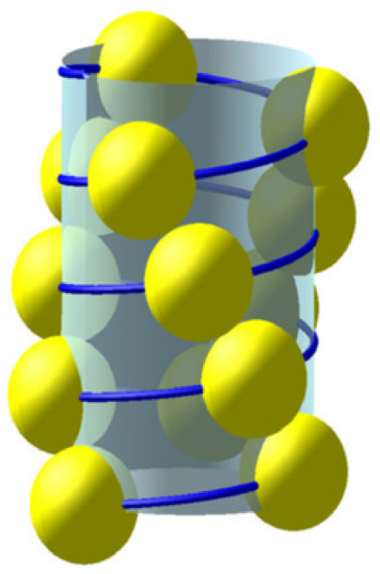
Draw a parallelogram around one of these buds...

We obtain the **fundamental parallelogram**:



Determining the **maximum size of a bud** on the cylindrical stem is equivalent to finding the **largest disk that can be inscribed within the fundamental parallelogram.**

The Lattice Model



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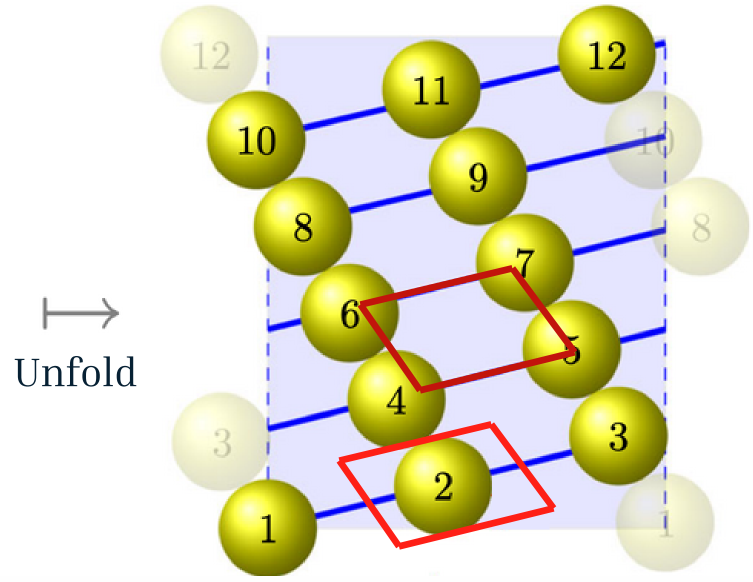

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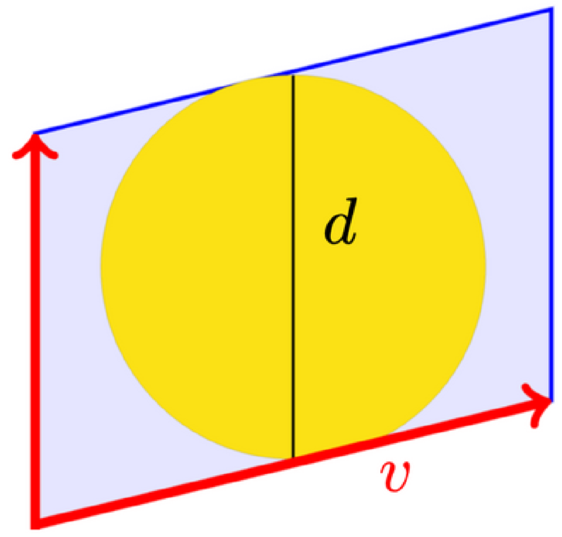

Disk inscribed
in the **fundamental parallelogram**

$$\text{A function on the lattice: } f^* = \frac{\text{Area of inscribed disk}}{\text{Area of parallelogram}}$$

$$f(\omega) := f^*(\mathbb{Z} + \mathbb{Z}\omega), \quad \omega = \theta + ih \in \mathcal{H}$$

The Growth Capacity Function

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$$\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

alternatively,

$$\mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}$$

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f is a **modular form**

f is **invariant** under the actions of $Sl_2(\mathbb{Z})$:

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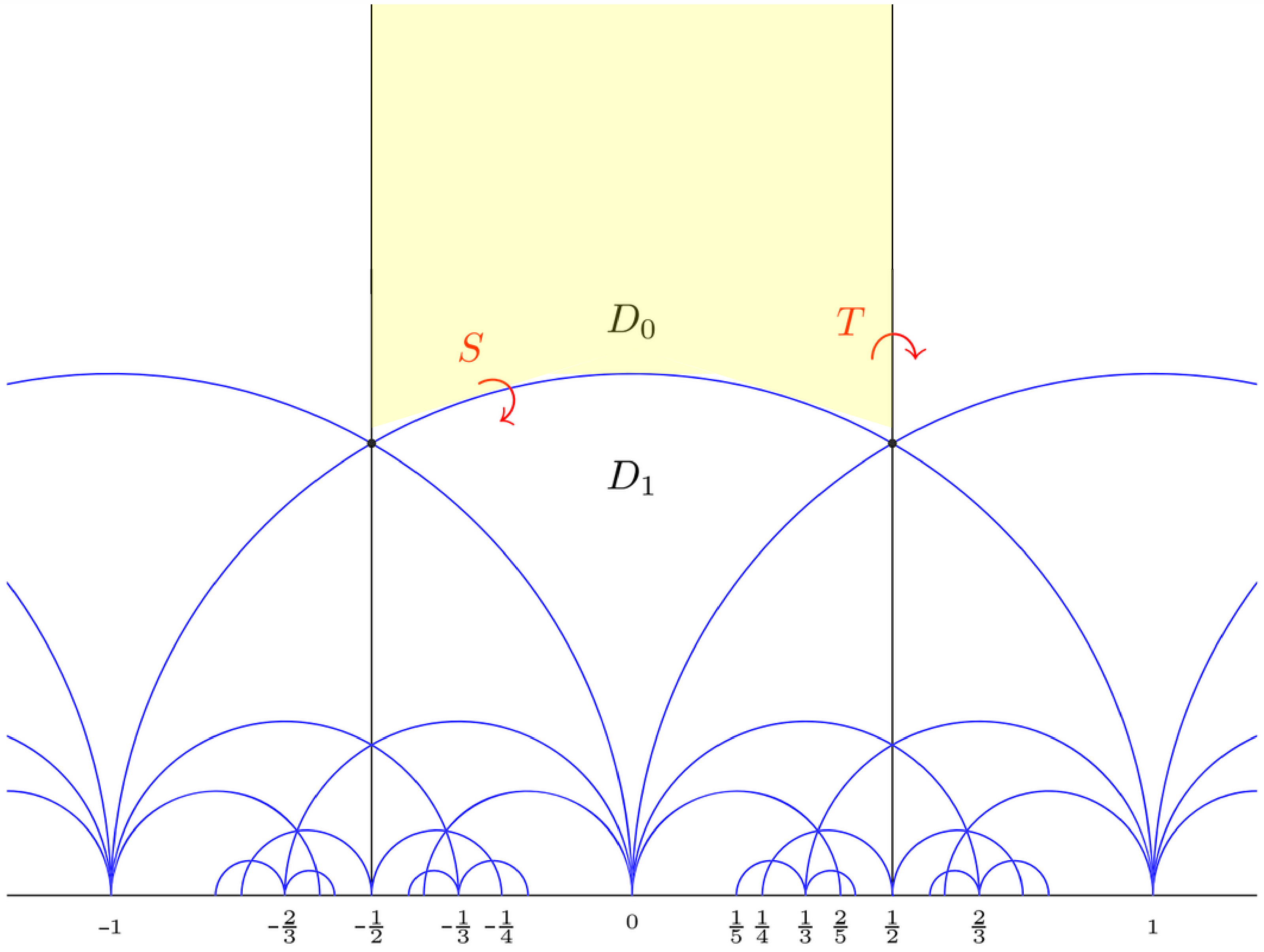
$SL_2(\mathbb{Z})$ is the **special linear group** on $\mathbb{Z}^2$

$$SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d, \in \mathbb{Z},\ ad - bc = 1 \right\}$$

The action of a matrix in $SL_2(\mathbb{Z})$ on $\omega \in \mathcal{H}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \omega = \frac{a\omega + b}{c\omega + d}$$

Explicit Formula



Tiling of hyperbolic plane

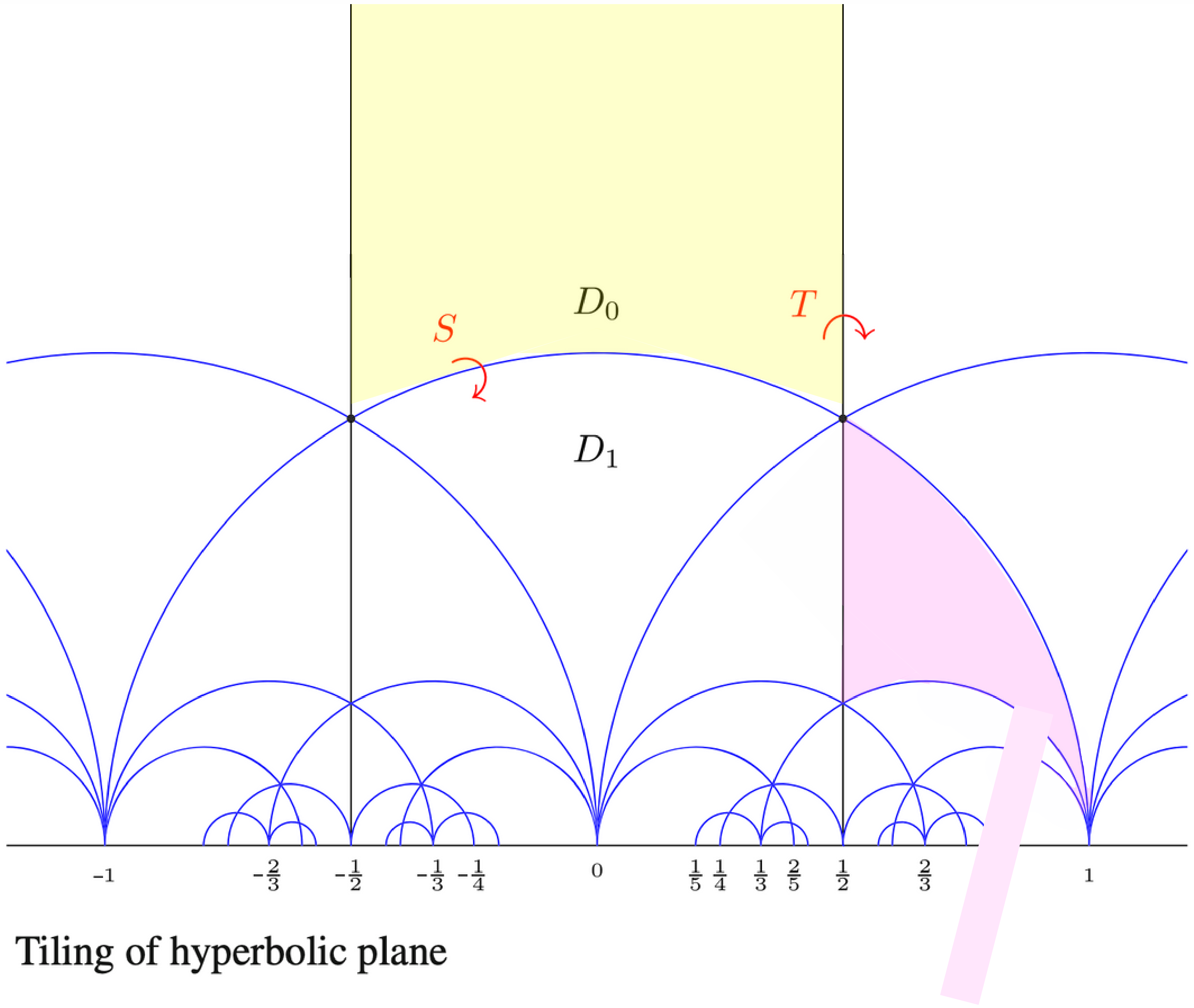
The **fundamental region**, $\mathcal{D}_0$, has the property:

For every $\omega \in \mathcal{H}$, there exist $\omega_0 \in \mathcal{D}_0$ and $A \in Sl_2(\mathbb{Z})$ such that

$$\omega = A\omega_0$$

In other words, it is **always possible** to be brought back to the fundamental region $\mathcal{D}_0$.

Explicit Formula



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a **transform** of D_0

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$$(f \text{ easy to compute on } \mathcal{D}_0) \wedge (f \text{ modular}) \implies \text{f is easily computable on all of } \mathcal{H}$$

$$\omega = \theta + ih \in \begin{bmatrix} p & r \\ q & s \end{bmatrix} \mathcal{D}_0 \implies f(\theta + ih) = \frac{\pi}{4}\left(\frac{(q\theta - p)^2}{h} + q^2 h\right), \quad \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in Sl_2(\mathbb{Z})$$

Key takeaway: for f to be large, you need $(q\theta - p)$ to be large.

Finale: The Golden Ratio

We obtain that

$$|q\theta - p| > \frac{C}{q} \implies f(\theta + ih) > \frac{C\pi}{2}, \text{ for some } h > 0$$

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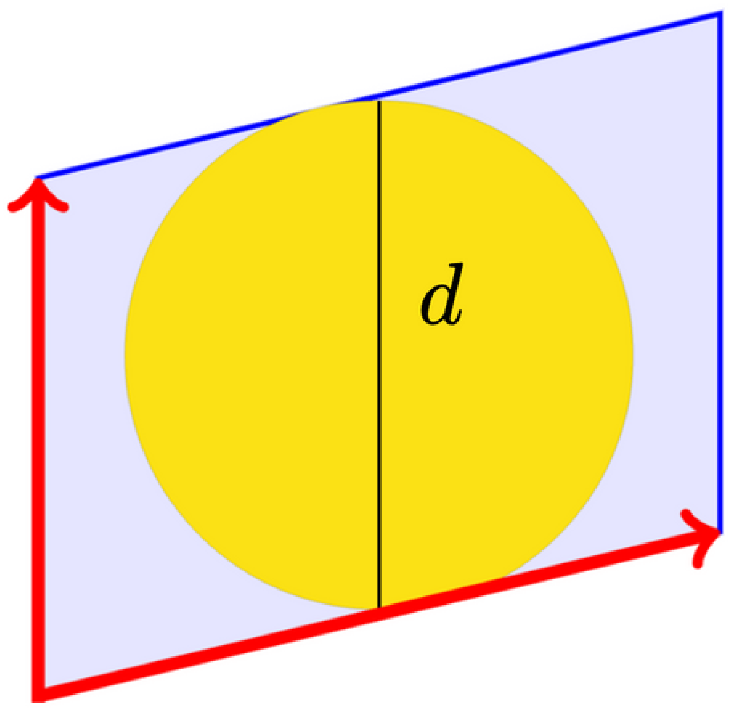
Using our knowledge of convergents, we deduce the following result:

The limit inferior as $h \to 0$ of $f(\theta + ih)$ is given by:

$$\liminf_{h \to 0} f(\theta + ih) = \begin{cases} \frac{\pi}{2\sqrt{5}} & \text{if } \theta = \frac{a\phi + b}{c\phi + d}, \; ad - bc = \pm 1 \\ \leq \frac{\pi}{2\sqrt{8}} & \text{otherwise} \end{cases}$$

Finale: The Golden Ratio - A Visualization



$$\frac{\pi}{2\sqrt{5}} \approx 0.702481 \approx 70.25\%$$

of the area of the parallelogram is the inscribed disk.

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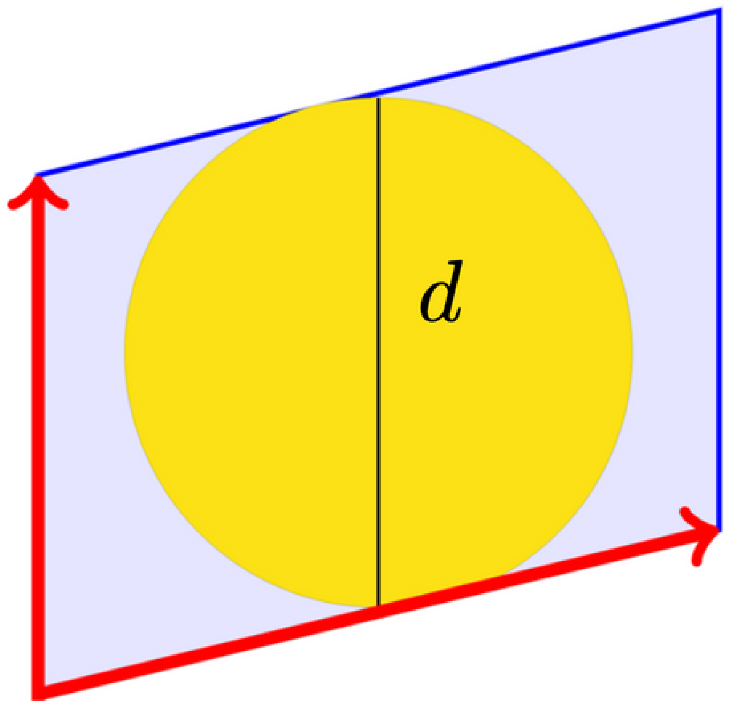

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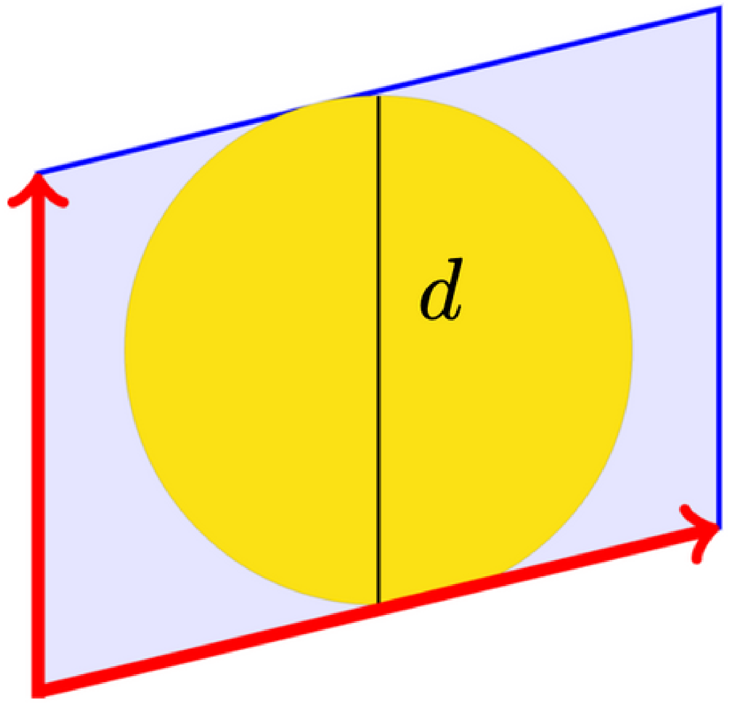

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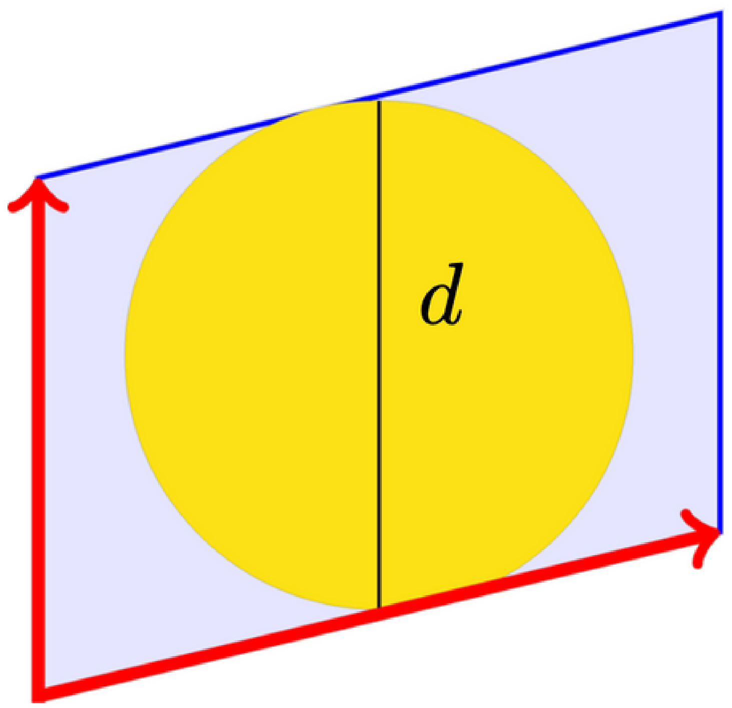

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When $\frac{a\phi + b}{c\phi + d}$, $ad - bc = \pm 1$, we have the **largest potential for growth**.

Thank You for Listening

Jackie Liu **Lydia He**

j385liu@uwaterloo.ca l32he@uwaterloo.ca