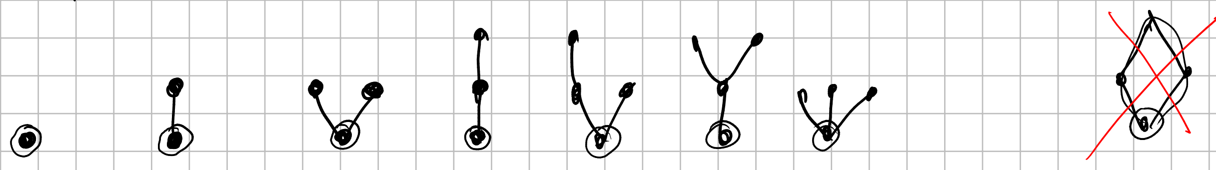
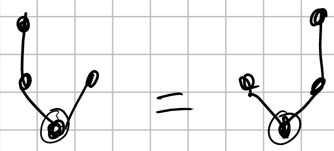
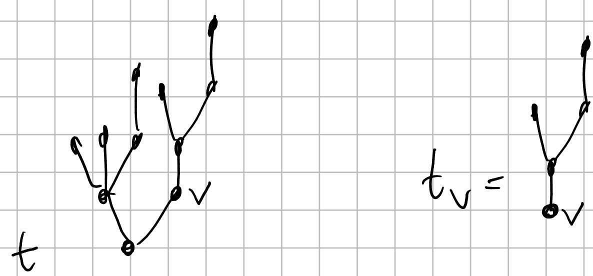
① Coproducts of rooted trees

eg 

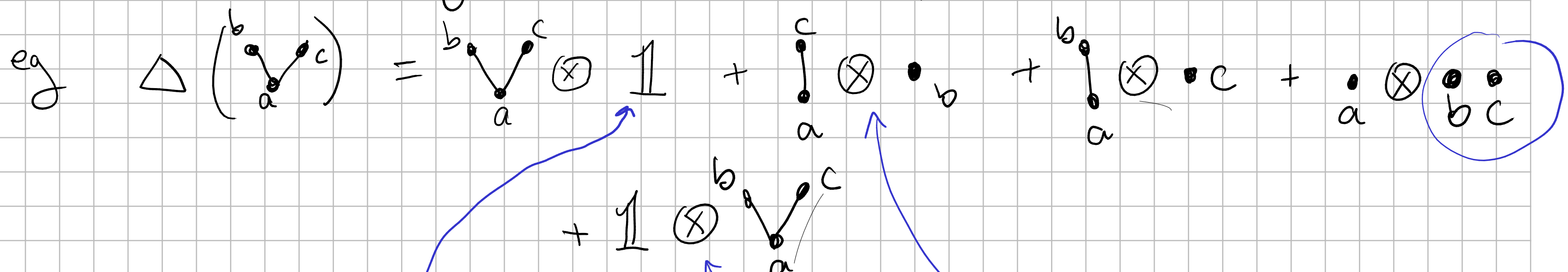
no order on children eg 

let t be a rooted tree, v a vertex of t

define t_v to be the subtree formed from v and v's children and v's children's children etc.

eg 

define a function telling us how to decompose rooted trees into pieces

eg 

use $\mathbb{1}$ for empty tree

use $\otimes$ to separate the parts for now – just treat it as a dividing line

In general

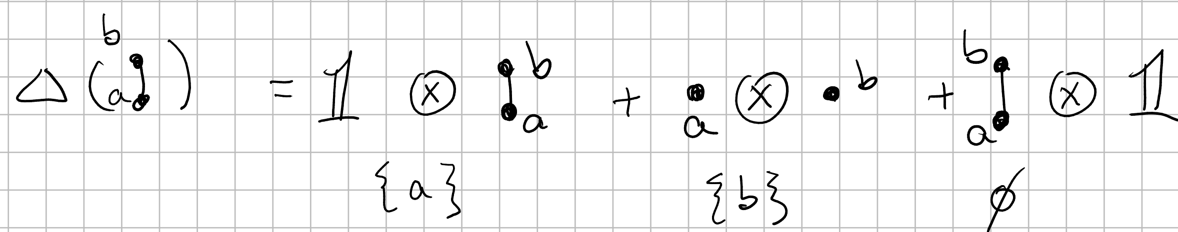
$$\Delta(t) = \sum_{S} \Big(t - \prod_{v \in S} t_v \Big) \otimes \Big(\prod_{v \in S} t_v \Big)$$

product is disjoint union
just write the trees next to each other

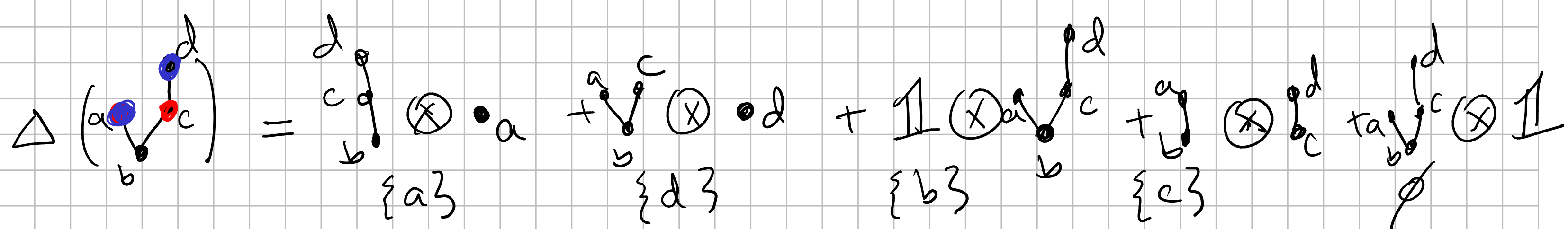
S subset of vertices of t so that no element of S is the <u>descendent</u> of another.

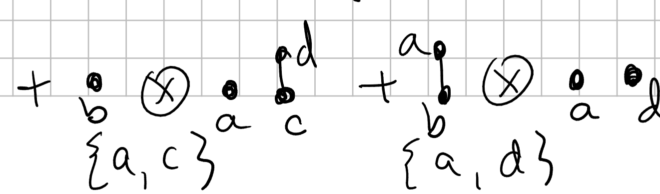
roots of parts we cut off

eg



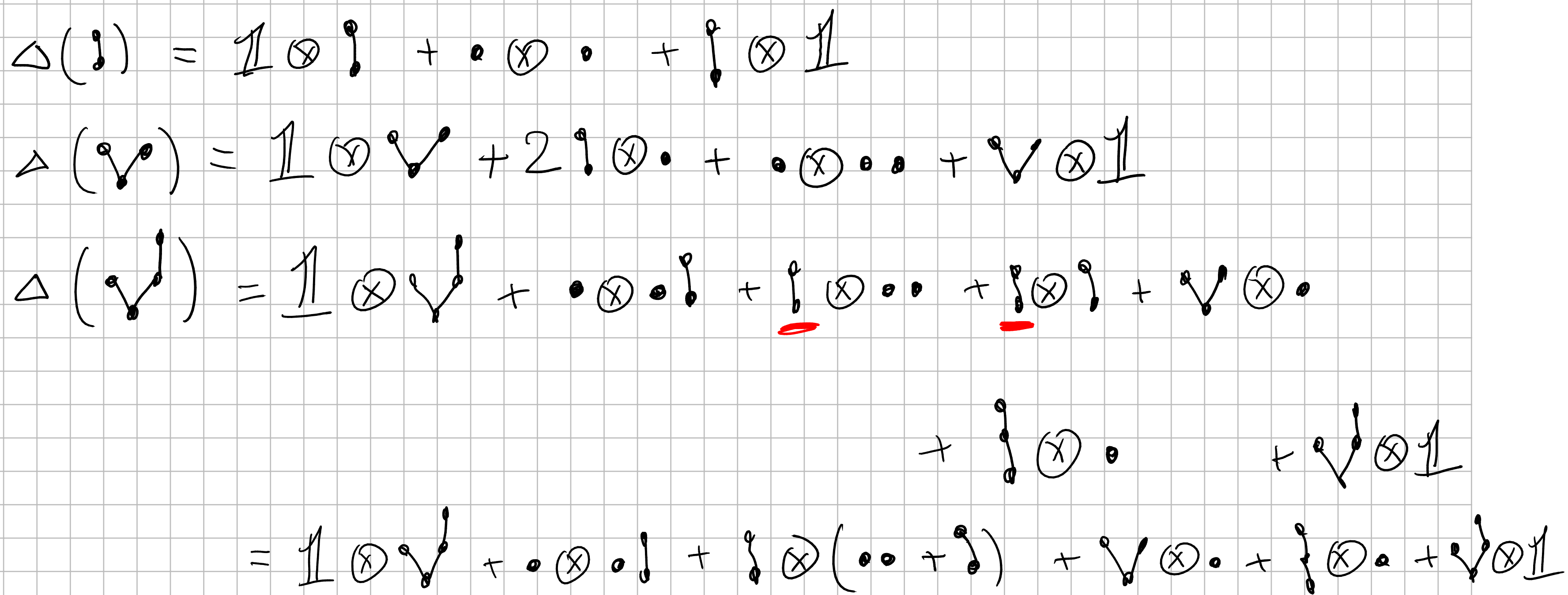
not $\{a,b\}$ because b is a child of a



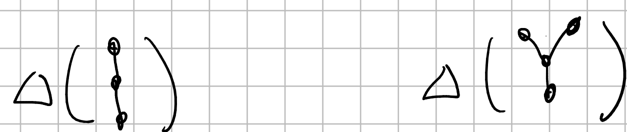


not $\{b,d\}$ because d is a descendent of b

Summarizing and removing the labels



You try $\Delta(\bullet)$ $\Delta(\mathbb{1})$



$$\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$$

$$\Delta(\bullet) = \mathbb{1} \otimes \bullet + \bullet \otimes \mathbb{1}$$

$$\Delta(\mathfrak{l}_3) = \mathbb{1} \otimes \mathfrak{l}_3 + \bullet \otimes \mathfrak{l}_2 + \mathfrak{l}_2 \otimes \bullet + \mathfrak{l}_3 \otimes \mathbb{1}$$

$$\Delta(Y_{a;b;c,d}) = Y \otimes \mathbb{1} + 2\,\mathfrak{l}_2 \otimes \bullet + \mathfrak{l}_2 \otimes \bullet\bullet + \bullet_a \otimes \vee_{b;c,d} + \mathbb{1} \otimes Y$$

② Closed under coproduct

Let $a_0 = \mathbb{1}$ $\qquad \Delta(a_0) = \mathbb{1} \otimes \mathbb{1} = a_0 \otimes a_0$

$a_1 = \bullet$ $\qquad \Delta(a_1) = a_0 \otimes a_1 + a_1 \otimes a_0$

$a_2 = \mathfrak{l}_2$ $\qquad \Delta(a_2) = a_0 \otimes a_2 + a_1 \otimes a_1 + a_2 \otimes a_0$

$a_3 = \mathfrak{l}_3 + \vee$ $\qquad \Delta(a_3) = \Delta(\mathfrak{l}_3) + \Delta(\vee)$

$$= \mathbb{1} \otimes \mathfrak{l}_3 + \bullet \otimes \mathfrak{l}_2 + \mathfrak{l}_2 \otimes \bullet + \mathfrak{l}_3 \otimes \mathbb{1}$$
$$+ \mathbb{1} \otimes \vee + 2\,\mathfrak{l}_2 \otimes \bullet + \bullet \otimes \bullet\bullet + \vee \otimes \mathbb{1}$$

$$= a_0 \otimes a_3 + 3a_2 \otimes a_1 + a_1 \otimes (a_2 + a_1^2) + a_3 \otimes a_0$$

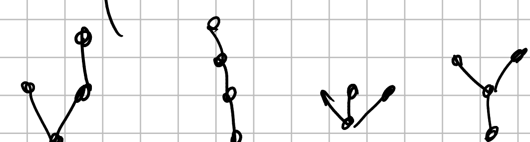
For you to think about

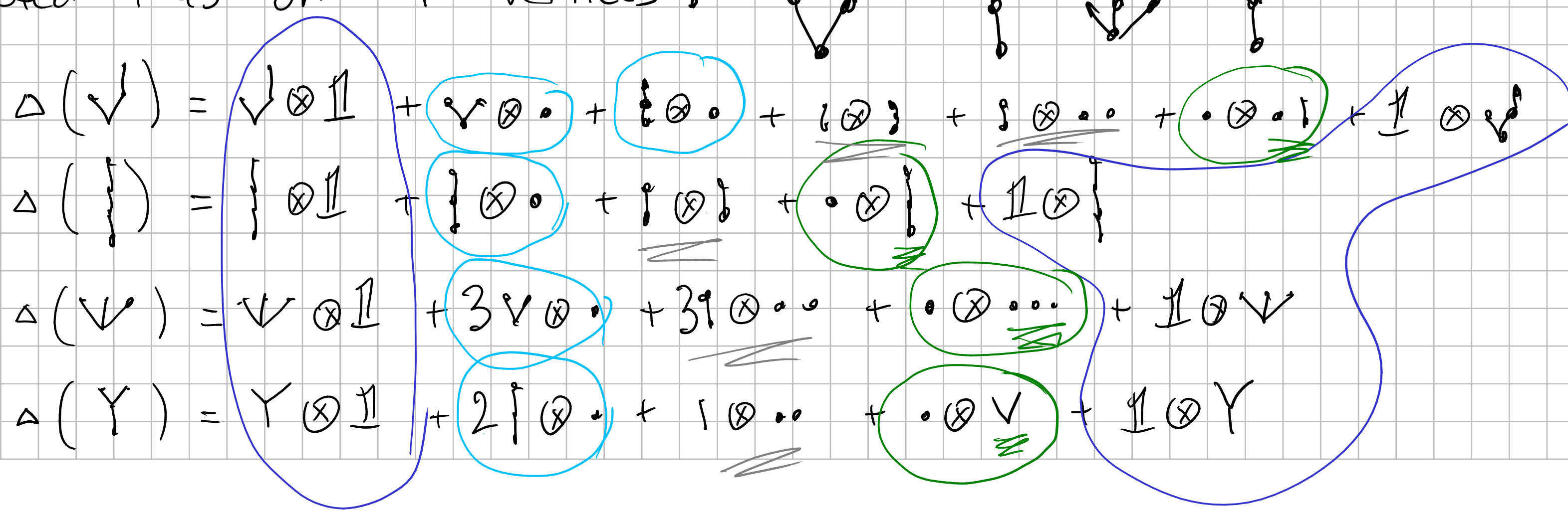
what can a_4 be so that I can write $\Delta(a_4)$ in terms of a_0, a_1, a_2, a_3, a_4 ?

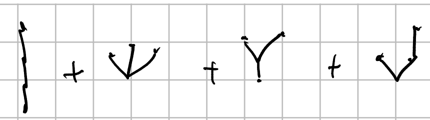
Hint $\rightsquigarrow$ might need a coefficient $\ldots 2$  $+ \ldots$

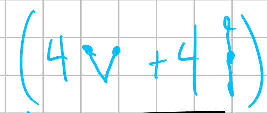
1st step $\rightsquigarrow$ find all rooted trees on 4 vertices and finish the catalog of coproducts.

rooted trees on 4 vertices:





$a_4 =$ 

$\Delta(a_4) = a_4 \otimes a_0 + a_0 \otimes a_4 +$  $\otimes a_1 + a_1 \otimes (a_3 + a_1^3 + a_1 a_2)$

$4a_3$

$+ 2a_2 \otimes a_2 + 5 a_2 \otimes a_1^2$

it still worked but not everything will because we need to get the previous pieces in the correct proportions

③ What was this really about?

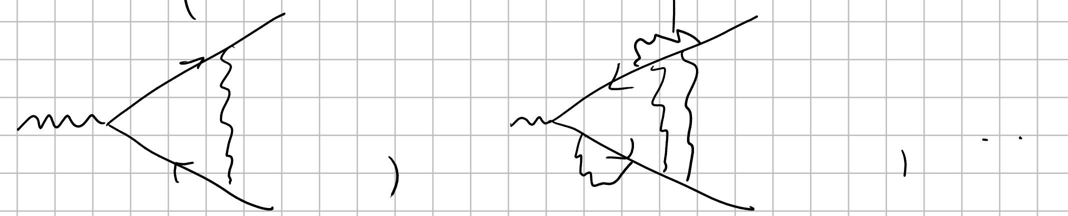
Framework $a_0 = 1$ $a_1 = \bullet$

$\dots$ a_i = some sum of trees of i vertices, potentially with coefficients

Then if $\Delta(a_i)$ can be written in terms of $a_0 \dots a_i$

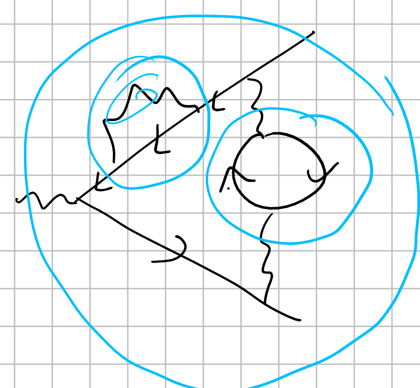
this gives the structure of a sub Hopf algebra

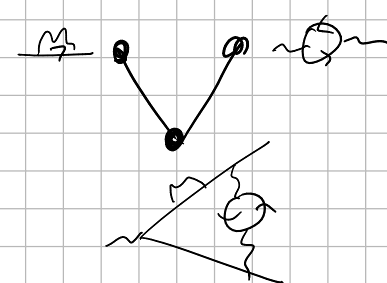
In perturbative quantum field theory care about possibilities for particle interactions



you particularly care about the structure of certain diagrams in the diagram

eg





so you care about rooted trees

but each diagram corresponds to a divergent integral

Use Δ to build a rule to fix the divergences.

Use sub Hopf algebras to fix them for a whole physics process.