The Graph Minor Theorem

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Directed reading program - Women in math

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Background

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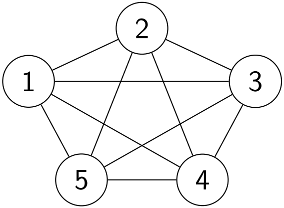
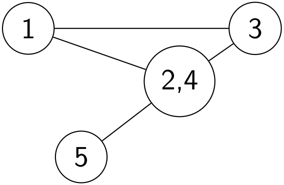
The "minor" is a relation on the class $\mathcal{G}$ of all graphs, denoted $\leq_m$.
So we have $\leq_m \subseteq \mathcal{G} \times \mathcal{G}$, so $\forall G, H \in \mathcal{G}, (H, G) \in \leq_m \iff H \leq_m G$.





Planar graphs

- A graph is **planar** if it can be drawn “nicely” on a plane.
- **Nicely**: edges do not cross except when sharing an end.
- This is the same as can be drawn nicely on a sphere.

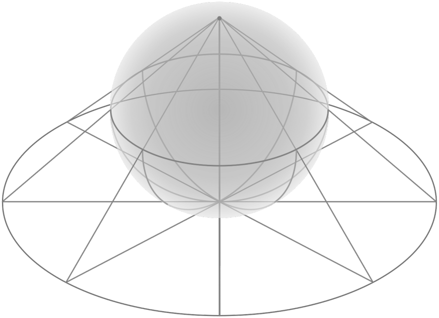


Figure: Plane = Sphere

Embeddings on other surfaces

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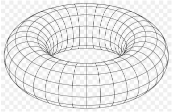

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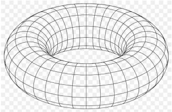

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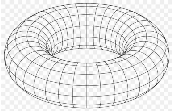

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Corollary

If S is a surface, G is a embeddable graph on S, and $H \leq_m G$, then H is also embeddable on S.

Partial orders, Forb's and a simple theorem

- Let $\leq_p$ be a partial order on A. We say a subset X of A is **closed** under $\leq_p$ if for every $x \in X$ and every $y \leq_p x$, we have $y \in X$.
- For every set A, **every** partial order $\leq_p$ on A and **every** subset X of A, we can define $Forb(X)$ as the set of all minimal elements of A which are **not** in X.

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Theorem

Let X be a subset of A which is closed under $\leq_p$, and let $a \in A$. Then we have $a \in X$ if and only if we have $b \not\leq_p a$ for all $b \in Forb(X)$.

The simple theorem stated for minors

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Theorem

Let $\mathcal{P}$ be a minor-closed class of graphs. Then a graph G belongs to $\mathcal{P}$ if and only if ***NO*** *graph in $Forb(\mathcal{P})$ is a minor of G.*

Planar graph: Kuratowski's Theorem

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For a surface S, we define $\mathcal{P}(S)$ as the class of all graphs embeddable on S.

Theorem

Kuratowski's Theorem 0.2 We have $Forb(\mathcal{P}(\mathrm{Sphere})) = \{K_5, K_{3,3}\}$.

The forbidden minor characterization

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Corollary

*Every **proper** minor-closed class $\mathcal{C}$ has non-empty Forb, and so has a Kuratowski-type characterization.*

For instance, a graph G is a forest if and only if K_3 is not a minor of G.

Lead to the no K_t minors graphs

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- Therefore, if we want to understand proper minor-closed classes, then we only need to understand the graphs with no K_t-minor, where t is a positive integer.
- Question: "What do graphs with no K_t minor look like?"

The Ever-startling Graph Minor Structure Theorem

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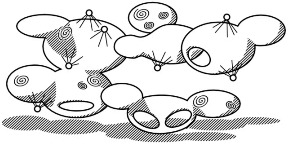
- The whole point of the graph minors project (by Robertson and Seymour) is an amazing realization: graphs on surfaces are so inseparably attached to minors and minor-closed classes.
- If we are talking about **any** minor-closed class, then we are always talking about graphs that are, **for the most part**, embeddable on some surface.

Theorem (Graph Minor Structure Theorem (VERY simplified))

For every positive integer t, there is a surface S such that every K_t-minor-free graph is (for the most part) embeddable on S.



Pictures by Felix Reidl

Well-quasi-order

Definition

A **well-quasi-order** $\leq_q$ on a set A is a partial order which does not have an infinite antichain.

The minor relation $\leq_m$ is a partial order on all graphs.

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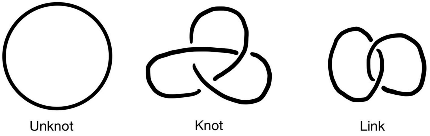
Idea of proof of Wagner's Conjecture. Let $G_1, G_2 \ldots$ be an infinite sequence of graphs. If $G_1 \leq_m G_i$ for some $i \geq 2$, then viola! Otherwise, $G_2, G_3 \ldots$ are all $K_{|G_1|}$-minor-free. Now the structure theorem is applied! (and the rest is 200+ pages long)

Applications to Topology

Consider the following properties. Given a graph G:

- When can we embed G in the 3-dimensional space with every cycle of G being **unknot**?
- When can we embed G in the 3-dimensional space such that no two cycles form a **non-splittable link**?



Both these properties are minor-closed! So by the graph minor theorem, they admit a Kuratowski-type characterization with a finite Forb set.

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- Allegedly when Donald Knuth saw this algorithm, he said: "well, maybe P is equal to NP!"

Any Questions?